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Departamento de Estatística e Investigación Operativa

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The Jackknife Estimate of Covariance under Censorship when Covariables are Present

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Abstract: In this paper the Jackknife estimate of covariance of two Kaplan-Meier integrals with covariates is introduced. Its strong consistency is established under mild conditions.

Keywords: strong consistency, survival analysis, censored data, Kaplan-Meier, reverse-time submartingale
MSC codes: 62G20, 62G09, and 62N02.

1 Introduction and main results

Let $Y_1, ..., Y_n$ be a sequence of lifetimes with distribution function F. In survival analysis, the lifetime Y may be observed exactly or may be known only up to a certain value. Along with the Y-sequence, let $C_1, ..., C_n$ be an independent sequence of random variables following a censoring distribution function G. So only the censored lifetimes $Z_i = min(Y_i, C_i)$ and $\delta_i = 1_{\{Y_i \leq C_i\}}$ are observable. Here δ_i is an indicator of whether Y_i has been observed or not. A nonparametric estimator of F based on the sample (Z_i, δ_i) , $i = 1, ..., n$, is given by the Kaplan-Meier estimator defined by:

$$1 - \hat{F}_n(t) = \prod_{i=1}^{n} [1 - \frac{\delta_{[i:n]}}{n-i+1}]^{1_{\{Z_{i:n} \leq t\}}}.$$

Here $Z_{1:n} \leq ... \leq Z_{n:n}$ are the ordered Z- values, where ties within life times or within censoring times are ordered arbitrarily and ties among lifetimes and censoring times are treated as if the former precedes the latter. $\delta_{[i:n]}$ is the concomitant of the ith-order statistic, that is, $\delta_{[i:n]} = \delta_j$ if $Z_{i:n} = Z_j$. But, often, in practice it occurs that, along with each (Z, δ), there is some additional information on the patient such as age, sex, blood pressure etc. In this case the available data are of the form (X_i, Z_i, δ_i), where X_i is a

p-variate vector of covariables paired with the possibly unobserved Y_i. To this end, Stute (1993) introduced an extension of the univariate Kaplan-Meier estimate to a multivariate setup, which is an estimator of the joint distribution function F^0 of (X, Y):

$$\hat{F}_n^0(x,y) = \sum_{i=1}^{n} W_{in} 1_{\{X_{[i:n]} \le x, Z_{i:n} \le y\}}$$

where for $i = 1, ..., n$

$$W_{in} = \frac{\delta_{[i:n]}}{n-i+1} \prod_{j=1}^{i-1} [\frac{n-j}{n-j+1}]^{\delta_{[j:n]}}$$

is the mass attached to $Z_{i:n}$ by $\hat{F}_n$, and $X_{[i:n]}$ is the i-th X-concomitant. Assume that φ is a Borel-measurable function from R^{p+1} into the real line. Set

$$S_n^{\varphi} = \sum_{i=1}^{n} W_{in} \varphi(X_{[i:n]}, Z_{[i:n]}) = \int \varphi d\hat{F}_n^0,$$

a so-called Kaplan-Meier integral.

Throughout this paper it will be assumed that the distribution function H of Z is continuous. Besides, for identifiability reasons we will assume

(i) Y and C are independent

(ii) $P(Y \le C|X, Y) = P(Y \le C|Y)$

Assumption (i) is standard in survival analysis, while (ii) states that, given the lifetime, the censoring indicator and the covariates are independent. Note that this assumption (ii) does not eliminate the possible dependences between the censoring time C and X. See Stute (1993, 1996a) for further discussion. Write

$$\tau_H = inf\{z : H(z) = 1\},$$

$$S^{\varphi} = \int_{\{Y \le \tau_H\}} \varphi dF^0.$$

Stute (1993) proved that, under (i) and (ii), with probability one and in the mean it holds

$$\lim_{n\to\infty} S_n^{\varphi} = S^{\varphi}. \tag{1.1}$$

Now, introduce the following sub-distribution functions:

$$\tilde{H}^{11}(x,z) = P(X \le x, Z \le z, \delta = 1),$$

$$\tilde{H}^0(z) = P(Z \le z, \delta = 0).$$

Under continuity we have:

$$S^\varphi = \int \varphi(x,w) \exp\{\int_0^w \frac{\tilde{H}^0(dz)}{1-H(z)}\} \tilde{H}^{11}(dx,dw).$$

For completely observable Y's, S_n^φ collapses to the sample mean of $\varphi(X_i, Y_i)$, $i = 1, ..., n$. Then, the central limit theorem states that, in distribution,

$$n^{1/2}[S_n^\varphi - S^\varphi] \to N(0, \sigma^2)$$

with

$$\sigma^2 = \int \varphi^2 dF^0 - [\int \varphi dF^0]^2.$$

In the general case, with censored data, the limit variance becomes much more complicated. To discuss the structure of the variance and the assumptions needed, set

$$\gamma_0(y) = \exp\{\int_0^{y^-} \frac{\tilde{H}^0(dz)}{1-H(z)}\},$$

$$\gamma_1^\varphi(y) = \frac{1}{1-H(y)} \int 1_{\{y<w\}} \varphi(x,w)\gamma_0(w)\tilde{H}^{11}(dx,dw),$$

and

$$\gamma_2^\varphi(y) = \int\int \frac{1_{\{v<y, v<w\}}\varphi(x,w)\gamma_0(w)}{[1-H(v)]^2} \tilde{H}^0(dv)\tilde{H}^{11}(dx,dw)$$

$$= \int 1_{\{v<y\}} \frac{\gamma_1^\varphi(v)}{1-H(v)} \tilde{H}^0(dv).$$

Now, consider the following conditions:
a) $\int [\varphi(X,Z)\gamma_0(Z)\delta]^2 dP < \infty$
b) $\int |\varphi(x,w)| q(K(w)) F^0(dx,dw) < \infty$, where $K(w) = \int_0^w \frac{G(dy)}{[1-H(y)][1-G(y)]}$
and q is a positive non-decreasing weight function on R^+
c) $\int q^{-2}(K) dK < \infty$

Under $(i), (ii)$ and $a) - c)$, in distribution,

$$n^{1/2}[S_n^\varphi - S^\varphi] \to N(0, \sigma^2)$$

where

$$\sigma^2 \equiv \sigma^2(\varphi) = Var\{\varphi(X,Z)\gamma_0(Z)\delta + \gamma_1^{\varphi}(Z)(1-\delta) - \gamma_2^{\varphi}(Z)\}.$$

See Theorem 1.1 in Stute (1996a). In this paper, we focus on the estimation of the limit variance σ^2. Stute (1996b) proved the consistency of the jackknife estimate of variance in the case without covariates. Here we extend his consistency result to the more general setting in which a vector of covariates is attached to each lifetime. Besides, in order to cover important applications such as e.g. variance estimation in multivariate (censored) linear regression (Stute 1993, 1996a), estimation of the covariance between two Kaplan-Meier integrals with covariates will be considered.

Specifically, to write the limit covariance (or variance, as a special case) in a simple way, introduce

$$\xi^{\varphi_i} = \varphi_i(X,Z)\gamma_0(Z)\delta + \gamma_1^{\varphi_i}(Z)(1-\delta) - \gamma_2^{\varphi_i}(Z)$$

and

$$\sigma_{ij} = Cov(\xi^{\varphi_i}, \xi^{\varphi_j}) = E(\xi^{\varphi_i}\xi^{\varphi_j}) - E(\xi^{\varphi_i})E(\xi^{\varphi_j}).$$

We have:

$$E\{\gamma_1^{\varphi_i}(Z)(1-\delta)\} = \int \gamma_1^{\varphi_i} d\tilde{H}^0 = \int\int_y^{\infty} \gamma_1^{\varphi_i}(y)\frac{H(dv)}{1-H(y)}\tilde{H}^0(dy)$$
$$= \int \gamma_2^{\varphi_i}(v)H(dv) = E\{\gamma_2^{\varphi_i}(Z)\}$$

and,

$$E\{\varphi_i(X,Z)\gamma_0(Z)\delta\gamma_2^{\varphi_j}(Z)\} = \int \varphi_i(x,y)\gamma_0(y)\gamma_2^{\varphi_j}(y)\tilde{H}^{11}(dx,dy)$$
$$= \int \varphi_i(x,y)\gamma_0(y)\int 1\{v<y\}\frac{\gamma_1^{\varphi_j}(v)}{1-H(v)}\tilde{H}^0(dv)\tilde{H}^{11}(dx,dy)$$
$$= \int \gamma_1^{\varphi_j}(v)\gamma_1^{\varphi_i}(v)\tilde{H}^0(dv)$$
$$= E\{\gamma_1^{\varphi_j}(Z)\gamma_1^{\varphi_i}(Z)(1-\delta)\}$$
$$= E\{\varphi_j(X,Z)\gamma_0(Z)\delta\gamma_2^{\varphi_i}(Z)\}.$$

We also have,

$$E\{\gamma_2^{\varphi_i}(Z)\gamma_2^{\varphi_j}(Z)\} = \int \gamma_2^{\varphi_i}(y)\gamma_2^{\varphi_j}(y)H(dy)$$

$$= \int\int\int 1\{y > max(u,v)\}\frac{\gamma_1^{\varphi_i}(u)}{1-H(u)}\frac{\gamma_1^{\varphi_j}(v)}{1-H(v)}H(dy)\tilde{H}^0(du)\tilde{H}^0(dv)$$

$$= \int \gamma_1^{\varphi_j}(v)\int 1\{u < v\}\frac{\gamma_1^{\varphi_i}(u)}{1-H(u)}\tilde{H}^0(du)\tilde{H}^0(dv)$$
$$+ \int \gamma_1^{\varphi_i}(u)\int 1\{v < u\}\frac{\gamma_1^{\varphi_j}(v)}{1-H(v)}\tilde{H}^0(dv)\tilde{H}^0(du)$$

$$= \int \gamma_1^{\varphi_j}(v)\gamma_2^{\varphi_i}(v)\tilde{H}^0(dv) + \int \gamma_1^{\varphi_i}(u)\gamma_2^{\varphi_j}(u)\tilde{H}^0(du)$$

$$= E\{\gamma_1^{\varphi_j}(Z)\gamma_2^{\varphi_i}(Z)(1-\delta)\} + E\{\gamma_1^{\varphi_i}(Z)\gamma_2^{\varphi_j}(Z)(1-\delta)\}.$$

Conclude that:

$$\sigma_{ij} = E\{\varphi_i(X,Z)\varphi_j(X,Z)\gamma_0^2(Z)\delta\} - E\{\gamma_1^{\varphi_j}(Z)\gamma_1^{\varphi_i}(Z)(1-\delta)\}$$
$$-E\{\varphi_i(X,Z)\gamma_0(Z)\delta\}E\{\varphi_j(X,Z)\gamma_0(Z)\delta\}.$$

In particular, for the asymptotic variance we have

$$\sigma^2 = E\{\varphi^2(X,Z)\gamma_0^2(Z)\delta\} - E\{(\gamma_1^{\varphi}(Z))^2(1-\delta)\} - E^2\{\varphi(X,Z)\gamma_0(Z)\delta\}.$$

Now, introduce the notation

$$S = S^{\varphi_1} = \int \varphi_1(x,w)\gamma_0(w)\tilde{H}^{11}(dx,dw) = E(\varphi_1(X,Z)\gamma_0(Z)\delta)$$

and

$$T = S^{\varphi_2} = \int \varphi_2(x,w)\gamma_0(w)\tilde{H}^{11}(dx,dw) = E(\varphi_2(X,Z)\gamma_0(Z)\delta),$$

so that

$$\sigma_{12} = E\{\varphi_1(X,Z)\varphi_2(X,Z)\gamma_0^2(Z)\delta\} - E\{\gamma_1^{\varphi_1}(Z)\gamma_1^{\varphi_2}(Z)(1-\delta)\} - S.T.$$

The jackknife estimator is based on the so-called pseudo values, which are obtained by computation of the estimator on the n leave-one-out subsamples of the initial dataset. For $k = 1, ..., n$, denote by $S_n^{\varphi(k)}$ the Kaplan-Meier integral with $(X_{[k:n]}, Z_{k:n}, \delta_{[k:n]})$ deleted from the sample. The pseudo values equal

$$S_n^{\varphi(k)} = \sum_{i=1}^{k-1} \frac{\varphi(X_{[i:n]}, Z_{i:n})\delta_{[i:n]}}{n-i} \prod_{j=1}^{i-1} (\frac{n-1-j}{n-j})^{\delta_{[j:n]}}$$

$$+ \sum_{i=k+1}^{n} \frac{\varphi(X_{[i:n]}, Z_{i:n})\delta_{[i:n]}}{n-i+1} \prod_{j=1}^{k-1} (\frac{n-1-j}{n-j})^{\delta_{[j:n]}} \prod_{j=k+1}^{i-1} (\frac{n-j}{n-j+1})^{\delta_{[j:n]}}$$

while the mean of the pseudo values is given by

$$\bar{S}_n^{\varphi} = S_n^{\varphi} - \varphi(X_{[n:n]}, Z_{n:n}) \frac{\delta_{[n:n]}(1-\delta_{[n-1:n]})}{n} \prod_{i=1}^{n-2} [\frac{n-i-1}{n-i}]^{\delta_{[i:n]}},$$

see Stute and Wang (1994). Set $S_n = S_n^{\varphi_1}$ and $T_n = S_n^{\varphi_2}$; $S_n^{(k)} = S_n^{\varphi_1(k)}$ and $T_n^{(k)} = S_n^{\varphi_2(k)}$. Also $\bar{S}_n = \bar{S}_n^{\varphi_1}, \bar{T}_n = \bar{S}_n^{\varphi_2}$.
The Jackknife estimate of covariance of S_n and T_n is now defined as

$$n\widehat{Cov}(Jack) = (n-1) \sum_{k=1}^{n} (S_n^{(k)} - \bar{S}_n)(T_n^{(k)} - \bar{T}_n)$$

$$= (n-1) \sum_{k=1}^{n} S_n^{(k)} T_n^{(k)} - n(n-1)\bar{S}_n \bar{T}_n$$

Hence, the calculation of a sum of cross products is needed. This is given in our first result.

Theorem 1.1: We have

$n\widehat{Cov}(Jack)$

$$= \{(n-1)n(\frac{n}{n-1})^{2\delta[1:n]}(\frac{n-2}{n})^{\delta_{[1:n]}} + (\delta_{[1:n]}-1)\frac{n}{n-1} - n(n-1)\}S_n T_n$$

$$+(n-1) \sum_{j=1}^{n-2} (\delta_{[j:n]}-1) b_j \prod_{k=1}^{j-1} [\frac{(n-k-1)(n-k+1)}{(n-k)(n-k)}]^{2\delta_{[k:n]}}$$

$$\times [\sum_{i=j+1}^{n} \varphi_1(X_{[i:n]}, Z_{i:n}) W_{in}].[\sum_{i=j+1}^{n} \varphi_2(X_{[i:n]}, Z_{i:n}) W_{in}]$$

$$+(n-1)\sum_{i=1}^{n-1}\varphi_1(X_{[i:n]},Z_{i:n})\varphi_2(X_{[i:n]},Z_{i:n})\delta_{[i:n]}\frac{1}{(n-i)^2}\prod_{j=1}^{i-1}[\frac{n-j-1}{n-j}]^{2\delta_{[j:n]}}$$

$$+(n-1)R_n\varphi_1(X_{[n:n]},Z_{n:n})\varphi_2(X_{[n:n]},Z_{n:n})W_{nn}^2$$

$$+(n-1)T_n\varphi_1(X_{[n:n]},Z_{n:n})\delta_{[n:n]}(1-\delta_{[n-1:n]})\prod_{i=1}^{n-2}[\frac{n-i-1}{n-i}]^{\delta_{[i:n]}}$$

$$+(n-1)S_n\varphi_2(X_{[n:n]},Z_{n:n})\delta_{[n:n]}(1-\delta_{[n-1:n]})\prod_{i=1}^{n-2}[\frac{n-i-1}{n-i}]^{\delta_{[i:n]}}$$

$$-\frac{n-1}{n}\varphi_1(X_{[n:n]},Z_{n:n})\varphi_2(X_{[n:n]},Z_{n:n})\delta_{[n:n]}(1-\delta_{[n-1:n]})\prod_{i=1}^{n-2}[\frac{n-i-1}{n-i}]^{2\delta_{[i:n]}}$$

where

$$b_j=b_{jn}=\frac{1}{(n-j-1)^2}-\frac{1}{(n-j)^2}+\frac{1}{(n-j-1)}-\frac{1}{(n-j)},$$

and

$$R_n=\frac{n}{(n-1)^2}-\sum_{j=1}^{n-2}(\delta_{[j:n]}-1)b_j\prod_{k=1}^{j-1}[\frac{(n-k-1)(n-k+1)}{(n-k)(n-k)}]^{2\delta_{[k:n]}}$$

$$+\sum_{j=1}^{n-1}\delta_{[j:n]}\frac{n-j+1}{(n-j)^3}\prod_{k=1}^{j-1}[\frac{(n-k-1)(n-k+1)}{(n-k)(n-k)}]^{2\delta_{[k:n]}}$$

$$-\prod_{k=1}^{n-1}[\frac{(n-k-1)(n-k+1)}{(n-k)(n-k)}]^{2\delta_{[k:n]}}.$$

The representation in Theorem 1.1 is crucial to prove the consistency of the Jackknife approach. It contains three main terms plus remainders. Each of these three terms corresponds to one of the terms for σ_{12} given above. To be precise, we will prove that the first term in Theorem 1.1 corresponds to $-S.T.$ On the other hand, it will be seen that the second and third terms in Theorem 1.1 converge to $-E\{\gamma_1^{\varphi_1}(Z)\gamma_1^{\varphi_2}(Z)(1-\delta)\}$ and $E\{\varphi_1(X,Z)\varphi_2(X,Z)\gamma_0^2(Z)\delta\}$ respectively. The term involving R_n will converge to zero. As regards the last three terms, they vanish asymptotically when both $\varphi_1(x,z)$ and $\varphi_2(x,z)$ vanish to the right of some z_0 strictly smaller than τ_H. For general φ_1 and φ_2 however, the proof is more involved. More specifically, the consistency of $n\widehat{Cov}(Jack)$ does not hold when $\delta_{[n-1:n]}=0$ and $\delta_{[n:n]}=1$. In this situation we artificially set

$\delta^*_{[n:n]} = 0$, and denote the modified estimate of covariance by $n\widehat{Cov}^*(Jack)$. Then the corresponding artificial Kaplan-Meier integrals S_n^* and T_n^* coincide with the mean of their pseudo values ($\bar{S}_n^*$ and $\bar{T}_n^*$ respectively) and consistency can be established.

To formally established the consistency of $\widehat{Cov}(Jack)$ a condition stronger than a) is required. So, assume

(iii) $\int \varphi_1(X,Z)\varphi_2(X,Z)(\gamma_0(Z)\delta)^2.\{-\ln(1-\sqrt{H(Z)})\}dP < \infty$

that is

$$\int \varphi_1(x,z)\varphi_2(x,z)\gamma_0^2(z).\{-\ln(1-\sqrt{H(z)})\}\tilde{H}^{11}(dx,dz). < \infty$$

As mentioned, we will also refer to the following support condition:

(iv) $\varphi_i(x,z) = 0$ for i=1,2 and $z > z_0$ for some $z_0 < \tau_H$.

Theorem 1.2. Under (iii), (iv) we have with probability 1

$$\lim_{n\to\infty} n\widehat{Cov}(Jack) = \sigma_{12}.$$

Theorem 1.3. Under (iii) we have with probability 1

$$\lim_{n\to\infty} n\widehat{Cov}^*(Jack) = \sigma_{12}.$$

We prove our main results in Section 2.

2 Proofs

Proof of Theorem 1.1. To prove Theorem 1.1 it is enough to calculate $\sum_{k=1}^{n} S_n^{(k)} T_n^{(k)}$. Because n remains fixed, to simplify the notation we omit it. Also the covariates play no role in the proof. Along this proof we will use the following form of notation:

$$Z_{(i)} \equiv Z_{i:n},\ \delta_{[i]} \equiv \delta_{[i:n]},\ W_i \equiv W_{in} = \frac{\delta_{[i]}}{n-i+1} \prod_{j=1}^{i-1} (\frac{n-j}{n-j+1})^{\delta_{[j]}}$$

and
$\varphi_1(Z_{(i)}) = \varphi_1(X_{[i:n]}, Z_{i:n})$, $\varphi_2(Z_{(i)}) = \varphi_2(X_{[i:n]}, Z_{i:n})$.

Then,
$S_n = \sum_{i=1}^{n} W_i \varphi_1(Z_{(i)})$, $T_n = \sum_{i=1}^{n} W_i \varphi_2(Z_{(i)})$

and

$$S_n^{(k)} = \sum_{i=1}^{k-1} \frac{\varphi_1(Z_{(i)})\delta_{[i]}}{n-i} \prod_{j=1}^{i-1} (\frac{n-1-j}{n-j})^{\delta_{[j]}}$$

$$+ \sum_{i=k+1}^{n} \frac{\varphi_1(Z_{(i)})\delta_{[i]}}{n-i+1} \prod_{j=1}^{k-1} (\frac{n-1-j}{n-j})^{\delta_{[j]}} \prod_{j=k+1}^{i-1} (\frac{n-j}{n-j+1})^{\delta_{[j]}} = S1^{(k)} + S2^{(k)},$$

$$T_n^{(k)} = \sum_{i=1}^{k-1} \frac{\varphi_2(Z_{(i)})\delta_{[i]}}{n-i} \prod_{j=1}^{i-1} (\frac{n-1-j}{n-j})^{\delta_{[j]}}$$

$$+ \sum_{i=k+1}^{n} \frac{\varphi_2(Z_{(i)})\delta_{[i]}}{n-i+1} \prod_{j=1}^{k-1} (\frac{n-1-j}{n-j})^{\delta_{[j]}} \prod_{j=k+1}^{i-1} (\frac{n-j}{n-j+1})^{\delta_{[j]}} = T1^{(k)} + T2^{(k)}$$

see Stute and Wang (1994) page 605. Therefore,

$$\sum_{k=1}^{n} S_n^{(k)} T_n^{(k)} = \sum_{k=1}^{n} S1^{(k)} T1^{(k)} + \sum_{k=1}^{n} S1^{(k)} T2^{(k)} + \sum_{k=1}^{n} S2^{(k)} T1^{(k)} + \sum_{k=1}^{n} S2^{(k)} T2^{(k)}.$$

Now, for $1 \leq i < r \leq n$ put

$$A_{ir} = \frac{1}{(n-i)} \prod_{j=1}^{i-1} [\frac{n-j-1}{n-j}]^{2\delta_{[j]}} \prod_{j=i}^{r-1} [\frac{n-j-1}{n-j}]^{\delta_{[j]}},$$

$$B_{ir} = \sum_{k=i+1}^{r-1} \frac{1}{(n-i)(n-r+1)}$$
$$\prod_{j=1}^{i-1} [\frac{n-j-1}{n-j}]^{2\delta_{[j]}} \prod_{j=i}^{k-1} [\frac{n-j-1}{n-j}]^{\delta_{[j]}} \prod_{j=k+1}^{r-1} [\frac{n-j}{n-j+1}]^{\delta_{[j]}},$$

$$C_{ir} = \sum_{k=1}^{i-1} \frac{1}{(n-i+1)(n-r+1)}$$
$$\times \prod_{j=1}^{k-1} [\frac{n-j-1}{n-j}]^{2\delta_{[j]}} \prod_{j=k+1}^{i-1} [\frac{n-j}{n-j+1}]^{2\delta_{[j]}} \prod_{j=i}^{r-1} [\frac{n-j}{n-j+1}]^{\delta_{[j]}},$$

$$D_i = \frac{1}{n-i} \prod_{j=1}^{i-1} [\frac{n-j-1}{n-j}]^{2\delta_{[j]}}\ , \ \ 1 \le i \le n-1,$$
$$E_i = \sum_{k=1}^{i-1} \frac{1}{(n-i+1)^2} \prod_{j=1}^{k-1} [\frac{n-j-1}{n-j}]^{2\delta_{[j]}} \prod_{j=k+1}^{i-1} [\frac{n-j}{n-j+1}]^{2\delta_{[j]}},$$
$2 \le i \le n$. Then, after some calculation we obtain,

$$\sum_{i=1}^{n} S1^{(k)} T1^{(k)} = \sum_{1 \le i < r \le n} \varphi_1(Z_{(i)}) \varphi_2(Z_{(r)}) \delta_{[i]} \delta_{[r]} A_{ir} + \sum_{1 \le i < r \le n} \varphi_1(Z_{(r)}) \varphi_2(Z_{(i)}) \delta_{[i]} \delta_{[r]} A_{ir}$$
$$+ \sum_{1 \le i < n} \varphi_1(Z_{(i)}) \varphi_2(Z_{(i)}) \delta_{[i]} D_i$$

where $A_{in} = 0$, by convention;

$$\sum_{k=1}^{n} S1^{(k)} T2^{(k)} = \sum_{1 \le i < r \le n} \varphi_1(Z_{(i)}) \varphi_2(Z_{(r)}) \delta_{[i]} \delta_{[r]} B_{ir}.$$

Similarly

$$\sum_{k=1}^{n} S2^{(k)} T1^{(k)} = \sum_{1 \le i < r \le n} \varphi_1(Z_{(r)}) \varphi_2(Z_{(i)}) \delta_{[i]} \delta_{[r]} B_{ir};$$

and finally

$$\sum_{k=1}^{n} S2^{(k)}T2^{(k)} = \sum_{1\le i<r\le n} \varphi_1(Z_{(i)})\varphi_2(Z_{(r)})\delta_{[i]}\delta_{[r]}C_{ir} + \sum_{1\le i<r\le n} \varphi_1(Z_{(r)})\varphi_2(Z_{(i)})\delta_{[i]}\delta_{[r]}C_{ir}$$

$$+ \sum_{1<i\le n} \varphi_1(Z_{(i)})\varphi_2(Z_{(i)})\delta_{[i]}E_i.$$

Altogether we have

$$\sum_{k=1}^{n} S_n^{(k)}T_n^{(k)} = \sum_{1\le i<r\le n} (\varphi_1(Z_{(i)})\varphi_2(Z_{(r)}) + \varphi_1(Z_{(r)})\varphi_2(Z_{(i)}))\delta_{[i]}\delta_{[r]}.(A_{ir}+B_{ir}+C_{ir})$$

$$+ \sum_{1<i<n} \varphi_1(Z_{(i)})\varphi_2(Z_{(i)})\delta_{[i]}.(D_i + E_i)$$

$$+\varphi_1(Z_{(1)})\varphi_2(Z_{(1)})\delta_{[1]}D_1 + \varphi_1(Z_n)\varphi_2(Z_{(n)})\delta_{[n]}E_n$$

$$= I + II + \varphi_1(Z_{(1)})\varphi_2(Z_{(1)})\delta_{[1]}D_1 + \varphi_1(Z_{(n)})\varphi_2(Z_{(n)})\delta_{[n]}E_n.$$

By following Corollary 2.2 and Lemma 2.3 in Stute (1996b) and by replacing $2\varphi(Z_{(i)})\varphi(Z_{(r)})$ by $(\varphi_1(Z_{(i)})\varphi_2(Z_{(r)}) + \varphi_1(Z_{(r)})\varphi_2(Z_{(i)}))$ in Lemma 2.4 in Stute (1996b), we obtain

$$I = \{n[\frac{n-2}{n}]^{\delta_{[1]}}[\frac{n}{n-1}]^{2\delta_{[1]}} + (\delta_{[1]}-1)\frac{n}{(n-1)^2}\} \sum_{i\neq r} \varphi_1(Z_{(i)})\varphi_2(Z_{(r)})W_iW_r$$

$$+ \sum_{j=1}^{n-2} (\delta_{[j]}-1)b_j \prod_{k=1}^{j-1} [\frac{(n-k+1)(n-k-1)}{(n-k)(n-k)}]^{2\delta_{[k]}} I_j$$

and

$$II = \{n[\frac{n-2}{n}]^{\delta_{[1]}}[\frac{n}{n-1}]^{2\delta_{[1]}} + (\delta_{[1]}-1)\frac{n}{(n-1)^2}\} \sum_{i=2}^{n-1} \varphi_1(Z_{(i)})\varphi_2(Z_{(i)})W_i^2$$

$$+ \sum_{j=1}^{n-2} (\delta_{[j]}-1)b_j \prod_{k=1}^{j-1} [\frac{(n-k+1)(n-k-1)}{(n-k)(n-k)}]^{2\delta_{[k]}} J_j$$

$$+\sum_{i=2}^{n-1}\varphi_1(Z_{(i)})\varphi_2(Z_{(i)})\delta_{[i]}\frac{1}{(n-i)^2}\prod_{j=1}^{i-1}[\frac{n-j-1}{n-j}]^{2\delta_{[j]}}$$

where

$$I_j=\sum_{j<i<r\le n}(\varphi_1(Z_{(i)})\varphi_2(Z_{(r)})+\varphi_1(Z_{(r)})\varphi_2(Z_{(i)}))W_iW_r$$

and

$$J_j=\sum_{i=j+1}^{n-1}\varphi_1(Z_{(i)})\varphi_2(Z_{(i)})W_i^2.$$

By collecting the terms it can be easily seen that

$$\sum_{k=1}^{n}S_n^{(k)}T_n^{(k)}=\{n[\frac{n-2}{n}]^{\delta_{[1]}}[\frac{n}{n-1}]^{2\delta_{[1]}}+(\delta_{[1]}-1)\frac{n}{(n-1)^2}\}S_nT_n$$

$$+\sum_{j=1}^{n-2}(\delta_{[j]}-1)b_j\prod_{k=1}^{j-1}[\frac{(n-k+1)(n-k-1)}{(n-k)(n-k)}]^{2\delta_{[k]}}$$

$$\times[\sum_{i=j+1}^{n}\varphi_1(Z_{(i)})W_i][\sum_{i=j+1}^{n}\varphi_2(Z_{(i)})W_i]$$

$$+\sum_{i=1}^{n-1}\varphi_1(Z_{(i)})\varphi_2(Z_{(i)})\delta_{[i]}\frac{1}{(n-i)^2}\prod_{j=1}^{i-1}[\frac{n-j-1}{n-j}]^{2\delta_{[j]}}$$

$$+P_n(Z_{(1)})+Q_n(Z_{(n)})$$

in which

$$P_n(Z_{(1)})=-\{n[\frac{n-2}{n}]^{\delta_{[1]}}[\frac{n}{n-1}]^{2\delta_{[1]}}+(\delta_{[1]}-1)\frac{n}{(n-1)^2}\}\varphi_1(Z_{(1)})\varphi_2(Z_{(1)})W_1^2$$

$$+\varphi_1(Z_{(1)})\varphi_2(Z_{(1)})\delta_{[1]}D_1-\varphi_1(Z_{(1)})\varphi_2(Z_{(1)})\delta_{[1]}\frac{1}{(n-1)^2}$$

and

$$Q_n(Z_{(n)})=-\{n[\frac{n-2}{n}]^{\delta_{[1]}}[\frac{n}{n-1}]^{2\delta_{[1]}}+(\delta_{[1]}-1)\frac{n}{(n-1)^2}\}\varphi_1(Z_{(n)})\varphi_2(Z_{(n)})W_n^2$$

$$-\sum_{j=1}^{n-2}(\delta_{[j]}-1)b_j\prod_{k=1}^{j-1}[\frac{(n-k+1)(n-k-1)}{(n-k)(n-k)}]^{2\delta_{[k]}}\varphi_1(Z_{(n)})\varphi_2(Z_{(n)})W_n^2$$
$$+\varphi_1(Z_{(n)})\varphi_2(Z_{(n)})\delta_n E_n.$$

Since for both $\delta_{[1]}=0$ and $\delta_{[1]}=1$, $P_n(Z_{(1)})=0$, using Lemma 2.5 in Stute (1996b) and rearranging terms complete the proof. □

To prove the consistency of $n\widehat{Cov}(Jack)$ we will compare the first three terms in the representation of Theorem 1.1 with those corresponding to σ_{12}. Let $\psi(x,z)=\varphi_1(x,z)\varphi_2(x,z)$, and assume $\psi(x,z)\geq 0$, without loss of generality. Here we switch the notation to use $Z_{i:n}$, $\delta_{[i:n]}$, $\varphi_j(X_{[i:n]},Z_{(i:n)})$, $j=1,2$, again. Set

$$\mathcal{F}_n=\sigma(Z_{i:n},X_{[i:n]},\delta_{[i:n]},1\leq i\leq n,Z_i,X_i,\delta_i,i>n)$$

and

$$U_n=(n-1)\sum_{i=1}^{n-1}\delta_{[i:n]}\psi(X_{[i:n]},Z_{i:n})(n-i)^{-2}\prod_{j=1}^{i-1}(\frac{n-j-1}{n-j})^{2\delta_{[j:n]}}$$

which is the third term in the representation of Theorem 1.1. It is easily seen that U_n is adapted to $\mathcal{F}_n$, and $\{\mathcal{F}_n\}_{n\geq 1}$ is a nonincreasing sequence of sigma algebras. Following Stute (1996b) we have

$$E\{U_n|\mathcal{F}_{n+1}\}=(n-1)\textstyle\sum_{i=1}^{n-1}\delta_{[i:n+1]}\psi(X_{[i:n+1]},Z_{i:n+1})V_{i,n+1}$$
$$+(n-1)\delta_{[n:n+1]}\psi(X_{[n:n+1]},Z_{n:n+1})\frac{1}{n+1}$$
$$\times\sum_{k=1}^{n-1}\prod_{j=1}^{k-1}[\frac{n-j-1}{n-j}]^{2\delta_{[j:n+1]}}\prod_{j=k+1}^{n-1}[\frac{n-j}{n-j+1}]^{2\delta_{[j:n+1]}}$$

where

$$V_{i,n+1}=\frac{n-i+1}{(n-i)^2(n+1)}\textstyle\prod_{j=1}^{i-1}[\frac{n-j-1}{n-j}]^{2\delta_{[j:n+1]}}$$
$$+\frac{1}{(n-i+1)^2(n+1)}\sum_{k=1}^{i-1}\prod_{j=1}^{k-1}[\frac{n-j-1}{n-j}]^{2\delta_{[j:n+1]}}\prod_{j=k+1}^{i-1}[\frac{n-j}{n-j+1}]^{2\delta_{[j:n+1]}}.$$

Lemma 2.1. We have

$$E\{U_n|\mathcal{F}_{n+1}\} \geq U_{n+1} - 3\delta_{[n:n+1]}\psi(X_{[n:n+1]}, Z_{n:n+1}) \prod_{j=1}^{n-1}(\frac{n-j}{n-j+1})^{2\delta_{[j:n+1]}} \tag{2.1}$$

Proof: Lemma 2.1 may be proved similarly to Corollary 2.8 in Stute (1996b), using Lemma 2.6 and Lemma 2.7, in Stute (1996b). □

Lemma 2.2. We have

$$\begin{aligned} &E\{n\psi(X_{[n:n]}, Z_{n:n})W_{nn}^2|\mathcal{F}_{n+1}\} \\ &\qquad \geq (n+1)\psi(X_{[n+1:n+1]}, Z_{n+1:n+1})W_{n+1n+1}^2 \\ &\qquad\quad -2\psi(X_{[n+1:n+1]}, Z_{n+1:n+1})W_{n+1n+1}^2 \end{aligned} \tag{2.2}$$

Proof: Just replace $\varphi^2(Z_{n:n})$ by $\psi(X_{[n:n]}, Z_{n:n})$ in the proof to Lemma 2.9 in Stute (1996b). □

Lemmas 2.1 and 2.2 state that the sequences $\{U_n\}_{n\geq1}$ and $\{n\psi(X_{[n:n]}, Z_{n:n})W_{nn}^2\}_{n\geq1}$ are slightly disturbed reverse-time submartingales with respect to $\{\mathcal{F}_n\}_{n\geq1}$. Lemma A in Stute (1996b) is a generalization of martingale convergence theorem, ensuring that, under appropriate conditions, such sequences converge with probability 1 (and in the mean) to a finite limit. To justify the applicability of Lemma A, the following Lemma is needed. Note that condition (b) below is just condition (iii) in Lemma A, Stute (1996b), while condition (a) ensures condition (iii) in the referred Lemma A and, in its turn, allows for the identification of the limit.

Lemma 2.3. Under (iii), we have

(a) With probability 1, $nE\{\psi(X_{[n:n]}, Z_{n:n})W_{nn}^2\} \to 0$

(b)$\sum_{n\geq 2} E(\psi(X_{[n:n]}, Z_{n:n})W_{nn}^2) < \infty$

Proof: As in the proof of Lemma 2.10 in Stute (1996b), without loss of generality, it can be assumed that Zs are uniformly distributed on $[0,1]$. Let $m(t) = P(\delta = 1|Z = t)$ and $\tilde{\phi}(z) = E(\psi(X,Z)\delta|Z = z)$.
Then,

$$nE(\psi(X_{[n:n]}, Z_{n:n})W_{nn}^2) = nE(\psi(X_{[n:n]}, Z_{n:n})\delta_{[n:n]} \prod_{i=1}^{n-1} [\frac{n-i}{n-i+1}]^{2\delta_{[i:n]}})$$

$$= nE(E(\psi(X_{[n:n]}, Z_{n:n})\delta_{[n:n]} \prod_{i=1}^{n-1} [\frac{n-i}{n-i+1}]^{2\delta_{[i:n]}} |Z_{1:n}, ..., Z_{n:n})).$$

By applying Lemma 2.1 in Stute (1993) and the distributional theory of uniform order statistics, we have

$$nE(\psi(X_{[n:n]}, Z_{n:n})W_{nn}^2)$$

$$= nE(\tilde{\phi}(Z_{n:n}) \prod_{i=1}^{n-1} [1 - \frac{2m(Z_{i:n})}{n-i+1} + \frac{m(Z_{i:n})}{(n-i+1)^2}])$$

$$= \frac{1}{n} E(\tilde{\phi}(Z_n)) 1_{\{Z_n = Z_{n:n}\}} \prod_{i=1}^{n-1} [1 + \frac{2(1-m(Z_i))}{n - R_{in}} + \frac{1-m(Z_i)}{(n-R_{in})^2}]^{1_{\{Z_i < Z_n\}}})$$

$$= \frac{1}{n} \int_0^1 \tilde{\phi}(u) E\{1_{\{Z_{n-1:n-1} \leq u\}} \prod_{i=1}^{n-1} [1 + \frac{2(1-m(Z_{i:n-1}))}{n-i} + \frac{1-m(Z_{i:n-1})}{(n-i)^2}]^{1_{\{Z_{i:n-1} < u\}}}\} du,$$

where R_{in} is the rank of Z_i among $Z_1, ..., Z_n$. By Lemma 2.13 of Stute (1996b) the inner product converges to $\gamma_0^2(u)$ with probability 1 for each $0 < u < 1$. If we use the dominated convergence argument and Chauchy-Schwarz inequality to bound the E-term, it can be shown that the last term is bounded from above, up to a constant factor, by

$$\frac{1}{n} \int_0^1 \tilde{\phi}(u)\gamma_0^2(u) u^{\frac{n-1}{2}} du.$$

Applying dominated convergence again implies that $nE\{n\psi(X_{[n:n]}, Z_{n:n})W_{nn}^2\} \to 0$ with probability 1, which is (a). And, since

$$\sum_{n\geq 1} \frac{z^{n/2}}{n} = -\ln(1 - \sqrt{z}),$$

condition (iii) yields (b). □

By (2.2), and Lemma 2.3(b), for $n\psi(X_{[n:n]}, Z_{n:n})W_{nn}^2$, it has been shown that the conditions of Lemma A in Stute (1996b) are met. So

$$\lim_{n\to\infty} n\psi(X_{[n:n]}, Z_{n:n})W_{nn}^2 = \lim_{n\to\infty} nE(\psi(X_{[n:n]}, Z_{n:n})W_{nn}^2) = 0.$$

Finally, because $R_n = O(1)$, we have with probability 1

$$nR_n\psi(X_{[n:n]}, Z_{n:n})W_{nn}^2 \to 0. \tag{2.3}$$

Now, we study the sequence $\{U_n\}_{n\geq 1}$. In order to show that U_n converges almost surely and in the mean to a finite limit, we should prove (cf Lemma A in Stute, 1996b)

$$\sum_{n\geq 2} E(\psi(X_{[n-1:n]}, Z_{n-1:n})\delta_{[n-1:n]}\prod_{j=1}^{n-2}[\frac{n-j-1}{n-j}]^{2\delta_{[j:n]}})$$

is finite. Before, we determine $E(U_n)$ in the following Lemma.

Lemma 2.4. Let $m(t) = P(\delta = 1|Z = t)$ and $\tilde{\phi}(z) = E(\psi(X, Z)\delta|Z = z)$.Then

$$E(U_n) = \frac{n}{n-1}E(\psi(X, Z)\delta g_{n-1}(Z))$$

where $g_n(t) = E(\phi_n(t))$ and

$$\phi_n(t) = \prod_{i=1}^{n-1}[1 + \frac{2(1-m(Z_{i:n}))}{n-i} + \frac{1-m(Z_{i:n})}{(n-i)^2}]^{1_{\{Z_{i:n}<t\}}}1_{\{t<Z_{n:n}\}}$$

Proof: By Lemma 2.1 of Stute (1993),

$$E(U_n) = (n-1)E(\sum_{i=1}^{n-1}\delta_{[i:n]}\psi(X_{[i:n]}, Z_{i:n})(n-i)^{-2}\prod_{j=1}^{i-1}[1 - \frac{\delta_{[j:n]}}{n-j}]^2)$$

$$= (n-1)E(\sum_{i=1}^{n-1}E(\delta_{[i:n]}\psi(X_{[i:n]}, Z_{i:n})(n-i)^{-2}\prod_{j=1}^{i-1}[1-\frac{2\delta_{[j:n]}}{n-j}+\frac{\delta_{[j:n]}}{(n-j)^2}]|Z_{1:n}, ..., Z_{n:n}))$$

$$= (n-1)E(\sum_{i=1}^{n-1}\tilde{\phi}(Z_{i:n})(n-i)^{-2}\prod_{j=1}^{i-1}(1-\frac{2m(Z_{j:n})}{n-j}+\frac{m(Z_{j:n})}{(n-j)^2}))$$

$$= \frac{1}{n-1}E(\sum_{i=1}^{n-1}\tilde{\phi}(Z_{i:n})\prod_{j=1}^{i-1}[1+\frac{2(1-m(Z_{j:n}))}{n-j-1}+\frac{1-m(Z_{j:n})}{(n-j-1)^2}])$$

$$= \frac{n}{n-1}E(\tilde{\phi}(Z_n)1_{\{Z_n<Z_{n:n}\}}\prod_{j=1}^{n}[1+\frac{2(1-m(Z_j))}{n-R_{jn}-1}+\frac{1-m(Z_j)}{(n-R_{jn}-1)^2}]^{1_{\{Z_j<Z_n\}}})$$

Since on $\{Z_j < Z_n\}$ we have $R_{jn} = R_{jn-1}$, also $Z_n < Z_{n:n}$ if and only if $Z_n < Z_{n-1:n-1}$, conditioning on Z_n completes the proof. □

Lemma 2.5. Under (iii)

$$\sum_{n\geq 2}E(\psi(X_{[n-1:n]}, Z_{n-1:n})\delta_{[n-1:n]}\prod_{j=1}^{n-2}[\frac{n-j-1}{n-j}]^{2\delta_{[j:n]}}) < \infty$$

Proof: The proof is similar to the proof of Lemma 2.3. Just note that

$$E(\psi(X_{[n-1:n]}, Z_{n-1:n})\delta_{[n-1:n]}\prod_{j=1}^{n-2}[\frac{n-j-1}{n-j}]^{2\delta_{[j:n]}})$$

$$= \frac{1}{(n-1)^2}\int_0^1\tilde{\phi}(u)E\{1_{\{Z_{n-2:n-1}\leq u\leq Z_{n-1:n-1}\}}$$

$$\times\prod_{j=1}^{n-2}[1+\frac{2(1-m(Z_{j:n-1}))}{n-j-1}+\frac{1-m(Z_{j:n-1})}{(n-j-1)^2}]^{1_{\{Z_{j:n-1}<u\}}}\}du$$

is bounded from above by

$$\frac{1}{n-1}\int_0^1\tilde{\phi}(u)\gamma_0^2(u)u^{\frac{n-2}{2}}du$$

. □

Corollary 2.6. Under (iii), with probability 1,

$$\lim_{n\to\infty}U_n = \lim_{n\to\infty}E(U_n) = E(\psi(X,Z)\gamma_0^2(Z)\delta).$$

Proof: Apply Lemma 2.13 and Lemma A in Stute (1996b), and Lemma 2.1 and Lemma 2.5 above. Then, by Hewitt-Savage 0-1 law, we get the result. □

We now study

$$II_{\varphi_1,\varphi_2} = (n-1)\sum_{j=1}^{n-2}(\delta_{[j:n]}-1)b_j\prod_{k=1}^{j-1}[\frac{(n-k-1)(n-k+1)}{(n-k)(n-k)}]^{2\delta_{[k:n]}}$$

$$\times[\sum_{i=j+1}^{n}\varphi_1(X_{[i:n]},Z_{i:n})W_{in}].[\sum_{i=j+1}^{n}\varphi_2(X_{[i:n]},Z_{i:n})W_{in}].$$

Lemma 2.7. Assume that φ_1,φ_2 satisfy the support condition (iv). Then, II_{φ_1,φ_2} converges to $-E\{\gamma_1^{\varphi_1}(Z)\gamma_1^{\varphi_2}(Z)(1-\delta)\}$ with probability 1.

Proof: Since under (iv) we have $\varphi_1(X_{[i:n]},Z_{i;n}) = \varphi_2(X_{[i:n]},Z_{i;n}) = 0$ at least for large i's (and large n), the summation may be restricted to $1 \le j \le n(1-\varepsilon)$ for some appropriate $0<\varepsilon<1$. Then, with probability 1, it can be seen that, the sum is asymptotically equivalent to

$$n\sum_{j=1}^{n}(\delta_{[j:n]}-1)(n-j+1)^{-2}[\sum_{i=j+1}^{n}\varphi_1(X_{[i:n]},Z_{i:n})W_{in}].[\sum_{i=j+1}^{n}\varphi_2(X_{[i:n]},Z_{i:n})W_{in}]$$

which can be written as:

$$-\int\frac{1}{(1-H_n(t^-))^2}[\int_{\{z>t\}}\varphi_1(x,z)\hat{F}_n^0(dx,dz)][\int_{\{z>t\}}\varphi_2(x,z)\hat{F}_n^0(dx,dz)]\tilde{H}_n^0(dt)$$

where $H_n(t) = \frac{1}{n}\sum_{i=1}^{n}1_{\{Z_i\le t\}}$ and $\tilde{H}_n^0(x) = \frac{1}{n}\sum_{i=1}^{n}1_{\{Z_i\le t,\delta_i=0\}}$. For each fixed t, and $i=1,2$, the Strong Law of Large Numbers for Kaplan-Meier integrals with covariates provides

$$\lim_{n\to\infty}\int_{\{z>t\}}\varphi_i(x,z)\hat{F}_n^0(dx,dz) = \int_{\{z>t\}}\varphi_i(x,z)\gamma_0(z)\tilde{H}^{11}(dx,dz)$$

with probability 1. By a standard Glivenko-Cantelli argument it can be shown that this convergence is uniform in t. In total , under condition (iv), with probability 1 it holds

$$\lim_{n\to\infty}\int \frac{1}{(1-H_n(t^-))^2}[\int_{\{z>t\}}\varphi_1(x,z)\hat{F}_n^0(dx,dz)][\int_{\{z>t\}}\varphi_2(x,z)\hat{F}_n^0(dx,dz)]\tilde{H}_n^0(dt)$$

$$=\int \gamma_1^{\varphi_1}(t)\gamma_1^{\varphi_2}(t)\tilde{H}^0(dt)$$

$$=E(\gamma_1^{\varphi_1}(Z)\gamma_1^{\varphi_2}(Z)(1-\delta)$$

. □

Proof of Theorem 1.2 . Since $Z_{n:n}\to\tau_H$ with probability 1, then with probability 1 there exists n_0 so that for $n>n_0$ we have $\varphi_1(X_{[n:n]},Z_{n:n})=0$,$\varphi_2(X_{[n:n]},Z_{n:n})=0$; also , $\bar{S}_n=S_n$, $\bar{T}_n=T_n$. So representation for $n\widehat{Cov}(Jack)$, in Theorem 1.1, reduces to the following formula

$$n\widehat{Cov}(Jack)$$

$$=\{(n-1)n(\frac{n}{n-1})^{2\delta[1:n]}(\frac{n-2}{n})^{\delta_{[1:n]}}+(\delta_{[1:n]}-1)\frac{n}{n-1}-n(n-1)\}S_nT_n$$

$$+(n-1)\sum_{j=1}^{n-2}(\delta_{[j:n]}-1)b_j\prod_{k=1}^{j-1}[\frac{(n-k-1)(n-k+1)}{(n-k)(n-k)}]^{2\delta_{[k:n]}}$$

$$\times[\sum_{i=j+1}^{n}\varphi_1(X_{[i:n]},Z_{i:n})W_{in}].[\sum_{i=j+1}^{n}\varphi_2(X_{[i:n]},Z_{i:n})W_{in}]$$

$$+(n-1)\sum_{i=1}^{n-1}\varphi_1(X_{[i:n]},Z_{i:n})\varphi_2(X_{[i:n]},Z_{i:n})\delta_{[i:n]}\frac{1}{(n-i)^2}\prod_{j=1}^{i-1}[\frac{n-j-1}{n-j}]^{2\delta_{[j:n]}}. \tag{2.4}$$

For both $\delta_{[1:n]}=0$, $\delta_{[1:n]}=1$, $\{...\}$ equals $\frac{-n}{n-1}$. Therefore, by applying the Strong Law of Large Numbers for Kaplan-Meier integrals with covariates (see (1.1)), the first term converges to $-S.T$ with probability 1. Under condition (iv) in Lemma 2.7 it has been proved that the middle sum, II_{φ_1,φ_2}, converges to $-E\{\gamma_1^{\varphi_1}(Z)\gamma_1^{\varphi_2}(Z)(1-\delta)\}$. Finally, the last sum , U_n, by Corollary 2.6 converges to $E\{\psi(X,Z)\gamma_0^2(Z)\delta\}$ with probability 1 as $n\to\infty$. □

Similar to the Jackknife estimate of variance of a Kaplan-Meier integral in Stute (1996b), when the support condition (iv) is not satisfied, the consistency of $\widehat{Cov}(Jack)$ is violated if $\delta_{[n-1:n]} = 0$ and $\delta_{[n:n]} = 1$. In this case, as mentioned, we introduce $\widehat{Cov}^*(Jack)$, a modified estimate of covariance with $\delta^*_{[n:n]} = 0$. We have

$$n\widehat{Cov}^*(Jack)$$

$$= \{(n-1)n(\frac{n}{n-1})^{2\delta[1:n]}(\frac{n-2}{n})^{\delta_{[1:n]}} + (\delta_{[1:n]}-1)\frac{n}{n-1} - n(n-1)\}S^*_nT^*_n$$

$$+(n-1)\sum_{j=1}^{n-2}(\delta_{[j:n]}-1)b_j\prod_{k=1}^{j-1}[\frac{(n-k-1)(n-k+1)}{(n-k)(n-k)}]^{2\delta_{[k:n]}}$$

$$\times[\sum_{i=j+1}^{n-1}\varphi_1(X_{[i:n]},Z_{i:n})W_{in}].[\sum_{i=j+1}^{n-1}\varphi_2(X_{[i:n]},Z_{i:n})W_{in}]$$

$$+(n-1)\sum_{i=1}^{n-1}\varphi_1(X_{[i:n]},Z_{i:n})\varphi_2(X_{[i:n]},Z_{i:n})\delta_{[i:n]}\frac{1}{(n-i)^2}\prod_{j=1}^{i-1}[\frac{n-j-1}{n-j}]^{2\delta_{[j:n]}}. \tag{2.5}$$

Since $Z_{n:n}$ converges to τ_H with probability 1, the application of the Strong Law of Large Numbers for Kaplan-Meier integrals with covariates indicates that
$\int \varphi_1(x,z)1_{\{Z_{n:n}=z\}}\hat{F}^0_n(dx,dz)$ converges to zero with probability 1. Therefore $|S^*_n - S_n|$ and $|T^*_n - T_n|$ are asymptotically negligible. Convergence of the first term and last sum in (2.5) hold for general φ_1 and φ_2, so to prove the consistency of $\widehat{Cov}^*(Jack)$ we will show that the convergence of the middle sum is not destroyed even if the support condition is not satisfied. To this end, consider $z_0 < \tau_H$ large enough and for $i = 1,2$ let $\varphi_i = \varphi_{i z_0} + \varphi_i^{z_0}$, where $\varphi_{i z_0}(x,z) = \varphi_i(x,z)1_{\{z\le z_0\}}$ and $\varphi_i^{z_0}(x,z) = \varphi_i(x,z)1_{\{z>z_0\}}$.
Now we rewrite the middle sum as

$$II^*_{\varphi_1,\varphi_2} = (n-1)\sum_{j=1}^{n-2}(\delta_{[j:n]}-1)b_j\prod_{k=1}^{j-1}[\frac{(n-k-1)(n-k+1)}{(n-k)(n-k)}]^{2\delta_{[k:n]}}$$

$$\times[\sum_{i=j+1}^{n-1}(\varphi_{1z_0}(X_{[i:n]},Z_{i:n})+\varphi_1^{z_0}(X_{[i:n]},Z_{i:n}))W_{in}]$$

$$.[\sum_{i=j+1}^{n-1}(\varphi_{2z_0}(X_{[i:n]},Z_{i:n})+\varphi_2^{z_0}(X_{[i:n]},Z_{i:n}))W_{in}]$$

$$=II^*_{\varphi_{1z_0},\varphi_{2z_0}}+II^*_{\varphi_{1z_0},\varphi_2^{z_0}}+II^*_{\varphi_1^{z_0},\varphi_{2z_0}}+II^*_{\varphi_1^{z_0},\varphi_2^{z_0}}.$$

Lemma 2.8. We have

$$\lim_{n\to\infty}II^*_{\varphi_{1z_0},\varphi_2^{z_0}}=\lim_{n\to\infty}II^*_{\varphi_1^{z_0},\varphi_{2z_0}}=0$$

Proof: We have that

$$II^*_{\varphi_{1z_0},\varphi_2^{z_0}}=(n-1)\sum_{j=1}^{n-2}(\delta_{[j:n]}-1)b_j\prod_{k=1}^{j-1}[\frac{(n-k-1)(n-k+1)}{(n-k)(n-k)}]^{2\delta_{[k:n]}}$$

$$\times[\sum_{i=j+1}^{n-1}\varphi_{1z_0}(X_{[i:n]},Z_{i:n})W_{in}].[\sum_{i=j+1}^{n-1}\varphi_2^{z_0}(X_{[i:n]},Z_{i:n})W_{in}]$$

is zero unless $\varphi_{1z_0}(X_{[i:n]},Z_{i:n})>0$ and $\varphi_2^{z_0}(X_{[k:n]},Z_{k:n})>0$ for some i and k , which implies $Z_{i:n}\le z_0$ and $Z_{k:n}>z_0$. In this case we must have $Z_{j+1}\le z_0$, that means we can reduce the summation to $1\le j\le n(1-\epsilon)$ for some appropriate $0<\epsilon<1$. Therefore, similarly to the proof of Lemma 2.7, $II^*_{\varphi_{1z_0},\varphi_2^{z_0}}$ is written as

$$\int\frac{-1}{(1-H_n(t^-))^2}[\int_{\{z>t\}}\varphi_{1z_0}(x,z)\hat{F}_n^0(dx,dz)][\int_{\{z>t\}}\varphi_2^{z_0}(x,z)\hat{F}_n^0(dx,dz)]\tilde{H}_n^0(dt)$$

and we may restrict the integration to $t's$ bounded away from τ_H. The Strong Law of Large Numbers for Kaplan-Meier integrals with covariates provides

$$\lim_{n\to\infty}\int_{\{z>t\}}\varphi_2^{z_0}(x,z)\hat{F}_n^0(dx,dz)=\int_{\{z>t\}}\varphi_2^{z_0}(x,z)\gamma_0(z)\tilde{H}^{11}(dx,dz)$$

$$=\int_{\{z>t\}}\varphi_2(x,z)1_{\{z>z_0\}}\gamma_0(z)\tilde{H}^{11}(dx,dz)$$

and since we chose z_0 large enough the limit is negligible and this completes the proof. □

Lemma 2.9.

$$\lim_{n\to\infty} II^*_{\varphi_1^{z_0},\varphi_2^{z_0}} = 0.$$

Proof: Without the loss of generality we consider $\varphi_1 \geq 0$ and $\varphi_2 \geq 0$. So $II^*_{\varphi_1^{z_0},\varphi_2^{z_0}}$ is not positive and it is sufficient to show that $II^*_{\varphi_1^{z_0},\varphi_2^{z_0}}$ is nonnegative asymptotically. Since for $i = 1, 2$, $\varphi_i^{z_0} \leq \varphi_1^{z_0} + \varphi_2^{z_0}$

$$II^*_{\varphi_1^{z_0},\varphi_2^{z_0}} \geq (n-1)\sum_{j=1}^{n-2}(\delta_{[j:n]}-1)b_j\prod_{k=1}^{j-1}[\frac{(n-k-1)(n-k+1)}{(n-k)(n-k)}]^{2\delta_{[k:n]}}$$

$$\times[\sum_{i=j+1}^{n-1}(\varphi_1^{z_0}(X_{[i:n]},Z_{i:n})+\varphi_2^{z_0}(X_{[i:n]},Z_{i:n}))W_{in}]^2.$$

Let $\phi = \varphi_1^{z_0} + \varphi_2^{z_0}$; by (2.5) we have (in obvious notation)

$n\widehat{Var}^*(Jack)$

$$= \{(n-1)n(\frac{n}{n-1})^{2\delta[1:n]}(\frac{n-2}{n})^{\delta_{[1:n]}} + (\delta_{[1:n]}-1)\frac{n}{n-1} - n(n-1)\}[S_n^{\phi^*}]^2$$

$$+(n-1)\sum_{j=1}^{n-2}(\delta_{[j:n]}-1)b_j\prod_{k=1}^{j-1}[\frac{(n-k-1)(n-k+1)}{(n-k)(n-k)}]^{2\delta_{[k:n]}}$$

$$\times[\sum_{i=j+1}^{n-1}\phi(X_{[i:n]},Z_{i:n})W_{in}]^2$$

$$+(n-1)\sum_{i=1}^{n-1}\phi^2(X_{[i:n]},Z_{i:n})\delta_{[i:n]}\frac{1}{(n-i)^2}\prod_{j=1}^{i-1}[\frac{n-j-1}{n-j}]^{2\delta_{[j:n]}}.$$

Since $S_n^{\phi^*} = \sum_{i=1}^{n-1} W_{in}\phi(X_{[i:n]}, Z_{[i:n]})$ converges to $S^\phi = \int_{\{Y\leq\tau_H\}}\phi dF^0$, and since the last sum converges to $E\{\phi^2(X,Z)\gamma_0^2(z)\delta\}$ with probability 1 (cfr. Corollary 2 .6 above), the non-negativity of $n\widehat{Var}^*(Jack)$ completes the proof. □

Proof of Theorem 1.3. As we have discussed before, the first term in (2.5) converges to $-S.T$, the last sum, U_n, does not contain $\delta_{[n:n]}$ and by Corollary 2.6 converges to $E\{\varphi_1(X,Z)\varphi_2(X,Z)\gamma_0^2(Z)\delta\}$ and along Lemmas 2.8 and 2.9 we showed that the middle sum, $II^*_{\varphi_1\varphi_2}$, is asymptotically equivalent to $II^*_{\varphi_{1_{z_0}}\varphi_{2_{z_0}}}$ where $\varphi_{1_{z_0}}, \varphi_{2_{z_0}}$ satisfy the support condition. Therefore the middle sum converges to $-E\{\gamma_1^{\varphi_1}(Z)\gamma_2^{\varphi_2}(Z)(1-\delta)\}$ by Lemma 2.7. □

Remark. As to the Jackknife estimate of variance of a Kaplan-Meier integral with covariates, the results of this paper hold. In such a case, in proofs we deal only with one Borel-measurable function, that is $\varphi_1(x,z) = \varphi_2(x,z) = \varphi(x,z)$.

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