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Report 10/03

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Departamento de Estatística e Investigación Operativa

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Estimation of a monotone percentile residual life function under random censorship

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Abstract

In this paper we introduce a new estimator of a percentile residual life function with censored data under a monotonicity constraint. Specifically, it is assumed that the percentile residual life is a decreasing function. We establish a law of the iterated logarithm for the proposed estimator, and its $\sqrt{n}$-equivalence to the unrestricted estimator (Chung, 1989). We investigate the finite sample performance of the monotone estimator in an extensive simulation study. Finally, a real data illustration is provided.

Key words and phrases: Aging notions, Censored data, Nonparametric estimation, Reliability, Survival Analysis

1 Introduction

The mean residual life is of interest in many areas of statistics and applied probability including biometry, actuarial science, and reliability. If the lifelength of a population is described by a random variable X, the mean residual life function at time t is defined to be the expected remaining life given survival up to time t.

In many applications it is reasonable to assume that the system life is monotonically degenerating or improving with age and Kochar et al. (2000) have studied the estimation of the mean residual life function under decreasing or increasing restrictions. Their estimator is a projection type estimator that proved to have nice properties in several restricted estimation problems; see, e.g. Rojo and Samaniego (1991, 1993), Mukerjee (1996), Rojo and Ma (1996), and Rojo (1995).

The mean residual life function is a useful tool for analyzing important properties of X when it exists because it characterizes the distribution. However, it has some weaknesses that may prevent its use. For example, it may not exist. Even when it exists it may have some practical shortcomings, especially in situations where the data are censored, or when the underlying distribution is skewed or heavy-tailed. In such cases, either the empirical mean residual life function cannot be calculated, or a single long-term survivor can have a marked effect upon it which will tend to be unstable due to its strong dependence on very long durations. Also, in an experiment it is often impossible or impractical to wait until all items have failed. In such cases, the median — or other percentiles — of the residual life of the random variable are useful alternatives to its mean residual life function.

Let X be a random variable, and let u be the right endpoint of its support. Let $\alpha \in (0, 1)$. For any $t < u$, the α-percentile residual life function at t, $q_\alpha(t)$, is defined as the α-percentile or quantile of the remaining life given survival up to time t. For $t \geq u$ we define $q_\alpha(t)$ to be zero. If F denotes the distribution function of X, then it holds

$$q_\alpha(t) = F^{-1}(\alpha + (1 - \alpha)F(t)) - t, \qquad t < u,$$

where $F^{-1}(p) = \inf\{x : F(x) \geq p\}$ is the so-called quantile function. Such a function describes, for example, the value that will be survived, by $(1-\alpha)\%$ of items (in reliability theory) or of individuals (in biology), among those that survived up to time t.

The α-percentile residual life functions were studied in some detail by Schmittlein and Morrison (1981), Arnold and Brockett (1983), Gupta and Langford (1984), Joe (1985), and more recently in Lin (2009). Families of distributions for which simple expressions for the α-percentile residual life functions can be obtained, are identified in Raja Rao, Alhumoud, and Damaraju (2006). A particular α-percentile residual life function of interest is the median residual life function given by $q_{0.5}$ that was studied in detail by Lillo (2005). In this paper the reader can find further references to papers that studied the α-percentile and the median residual life functions, and that used them in practical applications.

The estimation of q_α in the uncensored situation has been discussed by Csörgő and Csörgő (1987), Barabás et al. (1986), and Csörgő and Viharos (1992). In the estimation procedures considered by these authors, the empirical distribution function based on a random sample is used in place of the distribution function. When the data are randomly censored, the classical Kaplan-Meier product-limit estimator is used to obtain an estimator of q_α. See Csörgő (1987), and Chung (1989). More recently, Aly (1992) relaxed some conditions of the previous works for the construction of the confidence bands and introduced an alternative method based on bootstrap techniques. Kernel-type estimators were considered by Alam and Kulasekera (1993) in the uncensored situation. See also Feng and Kulasekera (1991) for the censored case.

Haines and Singpurwalla (1974), Joe and Proschan (1984a), and Franco-Pereira et al. (2011) studied some aspects of the classes of distribution functions with decreasing α-percentile residual life, $0 < \alpha < 1$. Besides, Franco-Pereira et al. (2011) initiated a study of the estimation of a percentile residual life function under monotone restrictions procedures following an approach that is similar to the approach of Kochar et al. (2000) in the uncensored situation. In this paper we extend their idea to the censored scenario and investigated the finite-sample behaviour of the new restricted estimator via simulations.

The rest of the paper is organized as follows. In Section 2 we introduce the restricted estimator and we give the main asymptotic results. In Section 3 we report a simulation study in which both the unrestricted and the restricted estimators are compared. A real data illustration is given in Section 4. Finally, main conclusions are reported in Section 5.

2 The estimator. Asymptotic results

Due to censoring, instead of the lifetime variable X one observes an i.i.d. sample (Z_1, δ_1), ..., (Z_n, δ_n) of the pair (Z, δ), where $Z = \min(X, C)$ is the observed time, $\delta = I(X \leq C)$ is the censoring indicator, and C is the potential censoring time. As usual, we assume that X and C are independent. In this setup, the nonparametric maximum-likelihood estimator of F is given by the Kaplan-Meier product-limit estimator

$$F_n(t) = 1 - \prod_{Z_{(i)} \leq t} \left[1 - \frac{\delta_{[i]}}{n-i+1}\right],$$

where $Z_{(1)} \leq ... \leq Z_{(n)}$ are the ordered Z-values, where ties within lifetimes or within censoring times are ordered arbitrarily, and ties among lifetimes and censoring times are treated as if the former precedes the later. Here, $\delta_{[i]}$ is the concomitant of the i-th ordered statistics, that is $\delta_{[i]} = \delta_j$ if $Z_{(i)} = Z_j$. Then, a natural nonparametric estimator of $q_\alpha(t)$ is defined as

$$\widehat{q}_\alpha(t) = F_n^{-1}\left(\alpha + (1-\alpha)F_n(t)\right) - t, \qquad t \leq Z_{(n)},$$

where $F_n^{-1}(p) = \inf\{x : F_n(x) \geq p\}$ stands for the empirical quantile associated to F_n. When $Z_{(n)}$ is uncensored, we have $F_n(Z_{(n)}) = 1$ and

$$\widehat{q}_\alpha(Z_{(n)}) = F_n^{-1}(1) - Z_{(n)} = 0.$$

In this case, the percentile residual life function $q_\alpha(t)$ is well-defined for all t. However, when the maximum observed time is censored, we have $F_n(Z_{(n)}) < 1$ and the value $\widehat{q}_\alpha(t)$ may not be well defined. More explicitly, the function $\widehat{q}_\alpha(t)$ is well defined for $t \leq \tau_n$ where

$$\tau_n = \sup\left\{x : F_n(x) \leq \frac{F_n(Z_{(n)}) - \alpha}{1-\alpha}\right\}.$$

Certainly, for $t \leq \tau_n$ we have $F_n(t) \leq \left(F_n(Z_{(n)}) - \alpha\right)/(1-\alpha)$ and hence $\alpha + (1-\alpha)F_n(t) \leq F_n(Z_{(n)})$, from which we have that the set $\Theta_t = \{x : F_n(x) \geq \alpha + (1-\alpha)F_n(t)\}$ is non-empty ($Z_{(n)}$ belongs to Θ_t). Therefore, $\widehat{q}_\alpha(t)$ exists for $t \leq \tau_n$. As n grows, we have $\tau_n \to \tau \equiv F^{-1}((1-\alpha)^{-1}(F(b_H) - \alpha))$, where b_H is the upper bound of the support of Z. In words: it is not possible to estimate consistently the percentile residual life function beyond time τ. This skips a portion of interest when b_H is smaller than the upper bound of the support of X. An analogous problem is found when recovering the cumulative distribution function $F(t)$ from the censored sample; in this case, consistency can not be obtained for $t > b_H$. In this sense, the almost sure and in-probability uniform rates in Theorems 1 and 2 below, which hold on an interval $[0, T]$ where $T < b_H \wedge \tau$, are almost the most one can expect in this scenario.

Throughout the paper we assume that $q_\alpha(t)$ is monotone decreasing. Then, we have $q_\alpha(t) = \inf_{y \leq t} q_\alpha(y)$ and a natural estimator of the percentile residual life function is introduced through

$$\widehat{q}^*_\alpha(t) = \inf_{y \leq t} \widehat{q}_\alpha(y).$$

Some asymptotic properties of $\widehat{q}^*_\alpha(t)$ are stated in the following results. Specifically, we establish a law of the iterated logarithm (LIL) and the $\sqrt{n}$-equivalence with respect to the unrestricted estimator.

Put H for the distribution function of Z and $b_H = \inf\{t : H(t) = 1\}$ for the upper limit of the support of Z. Let $T < b_H \wedge \tau$, i.e. $T < b_H$ and $F^{-1}(\alpha + (1-\alpha)F(T)) < b_H$. We refer to the following regularity conditions:

(C1) F is twice differentiable

(C2) $f = F'$ is bounded away from zero on $\left[F^{-1}(\alpha), F^{-1}(\alpha + (1-\alpha)F(T))\right]$

Theorem 1 (LIL) Under (C1) and (C2) we have w. p. 1

$$\sup_{0 \le t \le T} |\widehat{q}^*_\alpha(t) - q_\alpha(t)| = O\left(\left(\frac{\log\log n}{n}\right)^{1/2}\right).$$

Proof. The triangle inequality of the sup-norm (see Lemmas 1 and 2, Rojo and Samaniego, 1993), gives

$$\begin{aligned} |\widehat{q}^*_\alpha(t) - q_\alpha(t)| &= \left|\inf_{y \le t} \widehat{q}_\alpha(y) - \inf_{y \le t} q_\alpha(y)\right| \\ &\le \sup_{y \le t} |\widehat{q}_\alpha(y) - q_\alpha(y)|. \end{aligned}$$

Now, under the stated conditions, Theorem 8.1 in Chung (1989), see also Remark 8.1 in that paper, gives w. p. 1

$$\sup_{0 \le t \le T} |\widehat{q}_\alpha(t) - q_\alpha(t)| = O\left(\left(\frac{\log\log n}{n}\right)^{1/2}\right),$$

and the proof is complete.■

Now we establish the $\sqrt{n}$-equivalence between the restricted and the unrestricted estimators of $q_\alpha(t)$. From this second result, other asymptotic properties of the restricted estimator $\widehat{q}^*_\alpha(t)$ (e.g. weak convergence) may be automatically obtained from those of $\widehat{q}_\alpha(t)$.

Theorem 2. Assume that, with T as in Theorem 1,

(A1) $q'_\alpha(t)$ exists and $q'_\alpha(t) \le -c_1, 0 \le t \le T$, for some $c_1 > 0$

(A2) $q''_\alpha(t)$ exists and $\sup_{0 \le t \le T} \left|q''_\alpha(t)\right| \le c_2 < \infty$

(A3) Conditions (C1) and (C2) above hold

Then we have

$$\sqrt{n} \sup_{0 \le t \le T} |\widehat{q}^*_\alpha(t) - \widehat{q}_\alpha(t)| \to 0 \text{ in probability.}$$

Proof. The idea of the proof is that in Kochar et al. (2000). We first construct a continuous piecewise linear version of $\widehat{q}_\alpha(t)$, $L_n\widehat{q}_\alpha(t)$, on the interval $[0, T]$, and we show that it is eventually decresing with probability 1 (Lemma 1 below). For this, conditions (A1) and (A3) are needed; note that (A3) is just condition in Theorem 8.1 in Chung (1989) which, among other things, guarantees a LIL for $\widehat{q}_\alpha(t)$ (Remark 8.1, same paper). Then, we prove that $\widehat{q}_\alpha(t)$ and $\widehat{q}^*_\alpha(t)$ are close to $L_n\widehat{q}_\alpha(t)$ in an appropriate sense (Lemmas 2, 3, and 4 below).■

In order to introduce the piecewise linear version of $\widehat{q}_\alpha(t)$, for each n let k_n be an integer, $k_n \uparrow \infty$, and let $\Delta_n = T/k_n$. Let

$$a^n_j = jT/k_n = j\Delta_n, \qquad j = 0, 1, ..., k_n.$$

Define the linear interpolation of any function φ on $[0,T]$ by

$$L_n\varphi\left(a_j^n\right)=\varphi(a_j^n), \qquad j=0,1,...,k_n,$$

and, for $a_j^n<x<a_{j+1}^n$,

$$L_n\varphi(x)=\varphi(a_j^n)+\left[\varphi(a_{j+1}^n)-\varphi(a_j^n)\right](x-a_j^n)/\Delta_n, \qquad j=0,1,...,k_n-1.$$

Lemma 1. Under (A1) and (A3) we have $P\left[\underline{\lim}A_n\right]=1$ where

$$A_n=\left\{L_n\widehat{q}_\alpha(t) \text{ is strictly decreasing on } [0,b_H]\right\}.$$

Proof. Same lines as Proposition 4.1 in Kochar et al. (2000).■

Lemma 2. Under (A1) and (A3) we have $P\left[\underline{\lim}B_n\right]=1$ where

$$B_n=\left\{\sup_{0\le t\le T}\left|\widehat{q}_\alpha^*(t)-L_n\widehat{q}_\alpha(t)\right|\le\sup_{0\le t\le T}\left|\widehat{q}_\alpha(t)-L_n\widehat{q}_\alpha(t)\right|\right\}.$$

Proof. Same lines as Proposition 4.2 in Kochar et al. (2000). Note that by Lemma 1 we have a.s. for large n

$$L_n\widehat{q}_\alpha(t)=\inf_{y\le t}L_n\widehat{q}_\alpha(y),$$

and hence

$$\left|\widehat{q}_\alpha^*(t)-L_n\widehat{q}_\alpha(t)\right|=\left|\inf_{y\le t}\widehat{q}_\alpha(y)-\inf_{y\le t}L_n\widehat{q}_\alpha(y)\right|\le\sup_{y\le t}\left|\widehat{q}_\alpha(y)-L_n\widehat{q}_\alpha(y)\right|.■$$

Lemma 3. Under (A2) we have

$$\sup_{0\le t\le T}\left|q_\alpha(t)-L_nq_\alpha(t)\right|\le c_2\Delta_n^2.$$

Proof. Same lines as Proposition 4.3 in Kochar et al. (2000).■

Lemma 4. Under (A1), (A2), and (A3), if $n^{1/4}=o(k_n)$, we have

$$\sqrt{n}\sup_{0\le t\le T}\left|\widehat{q}_\alpha(t)-L_n\widehat{q}_\alpha(t)\right|\to 0 \qquad \text{in probability.}$$

Proof. Same lines as Proposition 4.4 in Kochar et al. (2000). The convergence of the percentile residual time process to a Gaussian process follows by Theorem 8.1 in Chung (1989). This is essential for bounding the term $\sqrt{n}\sup\left|\widehat{q}_\alpha(t)-V_nq_\alpha(t)\right|$ in that proof. The order on k_n is needed to bound $\sqrt{n}\sup\left|q_\alpha(t)-L_nq_\alpha(t)\right|$.■

3 Simulation study

In this Section we investigate the finite sample relative performance of the restricted and the unrestricted estimators $\widehat{q}^*_\alpha(t)$ and $\widehat{q}_\alpha(t)$ through simulations. For this, biases, variances, and mean squared errors (MSEs) of the estimators along the simulations are computed. The X variable is generated according to a Weibull distribution with shape and scale parameters 2 and 1 respectively, this is $F(t) = 1 - \exp(-t^2)$, $t \geq 0$. Note that the Weibull distribution has a decreasing α-percentile residual life function for all $\alpha \in (0, 1)$, when the shape parameter is larger than 1. The censoring variable is generated from a Weibull distribution with shape parameter 2 and scale parameter $\lambda = 0.188$, 0.479, 1.481, to get censoring percentages on X of about 15, 33 and 67%. The uncensored situation is also considered for comparison purposes. 10,000 Monte Carlo trials with sample sizes $n = 100, 250$ are generated. We consider the cases $\alpha = 0.25, 0.5, 0.75$, which correspond to the three quartile residual lifetime functions. We evaluate the estimators $\widehat{q}^*_\alpha(t)$ and $\widehat{q}_\alpha(t)$ at the values of t corresponding to the nine deciles of X.

In Figure 1 we depict the bias of both estimators along the nine deciles and the several censoring degrees, for the case $n = 100$ and $\alpha = 0.5$ (other cases report similar results). We see that the bias of the monotone estimator is negative, and that its absolute value is much larger than that of the unrestricted estimator; similar features are appreciated when e.g. estimating a monotone mean residual life function (Kochar et al., 2000). On the other hand, the unrestricted estimator shows a positive bias for most of the deciles. The absolute bias of both estimators increases when we move towards the right tail of X, something that is much more evident in the heavily censored case (67% of censoring). Besides, the huge bias of the monotone estimator (getting worse for larger deciles) is appreciated.

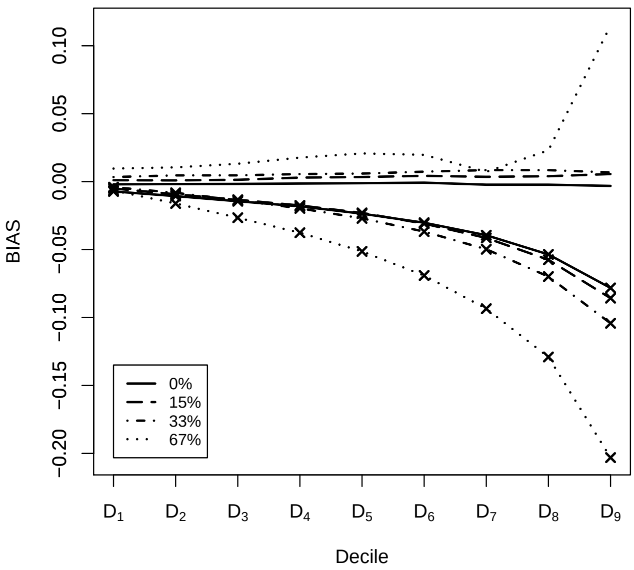

Figure 1: Bias of $\hat{q}_{0.5}$ and $\hat{q}^*_{0.5}$ (crosses) along the nine deciles, for different censoring degrees

In Table 1 we report the bias and the MSE of both estimators, $n = 100$ and $\alpha = 0.5$, for the several censoring levels and deciles 1, 2, 5, 8, and 9. Results corresponding to $n = 250$ are given in Table 2. From these Tables we can appreciate that the bias and the MSE go down when increasing the sample size; greater MSEs are obtained when working under heavier censoring levels. Besides, the MSEs of the unrestricted and the restricted estimators are closer to each other in the case $n = 250$, in agreement with the asymptotic equivalence stated in Section 2. The variance of the unrestricted estimator is much larger than that of the restricted one, and this compensates the excess in bias of the later. Indeed, the MSEs reported by the monotone estimator (which are greatly influenced by the bias term at the right tail of X) are less than the MSEs of the unrestricted estimator (mainly determined by the variance) in most of the cases. An exception to this is the uncensored situation, for which the variance of the unrestricted estimator is moderate and, as a consequence, it gives a better performance than the monotone estimator for some of the deciles. The other values of n and α report similar results (not shown).

Table 1: Bias and MSE of $\hat{q}_{0.5}$ and $\hat{q}^*_{0.5}$ for $n = 100$, and for different levels of censoring and deciles

		BIAS		MSE	
		$\hat{q}$	$\hat{q}^*$	$\hat{q}$	$\hat{q}^*$
	D_1	-0.00197	-0.00720	0.00347	0.00345
	D_2	-0.00180	-0.01065	0.00341	0.00334
$CP = 0\%$	D_5	-0.00115	-0.02358	0.00358	0.00342
	D_8	-0.00225	-0.05357	0.00550	0.00557
	D_9	-0.00315	-0.07820	0.00862	0.00888
	D_1	0.00104	-0.00503	0.00388	0.00382
	D_2	0.00090	-0.00930	0.00393	0.00378
$CP = 15\%$	D_5	0.00339	-0.02301	0.00444	0.00396
	D_8	0.00397	-0.05750	0.00775	0.00672
	D_9	0.00559	-0.08573	0.01393	0.01083
	D_1	0.00338	-0.00408	0.00491	0.00469
	D_2	0.00460	-0.00824	0.00502	0.00464
$CP = 33\%$	D_5	0.00593	-0.02703	0.00627	0.00511
	D_8	0.00844	-0.06984	0.01301	0.00932
	D_9	0.00683	-0.10421	0.02331	0.01545
	D_1	0.00960	-0.00657	0.01138	0.00948
	D_2	0.01049	-0.01608	0.01299	0.00972
$CP = 67\%$	D_5	0.02074	-0.05134	0.02344	0.01224
	D_8	-0.02303	-0.12900	0.03781	0.02692
	D_9	-0.11489	-0.20303	0.06173	0.05711

Figure 2 shows the MSEs of both estimators with respect to the censoring level, for three different deciles, and the case $n = 100$ and $\alpha = 0.5$. We see that the error gets worse when increasing the censoring degree. It is also seen that the MSE of the monotone estimator is lower that that of the unrestricted one (as discussed). Finally, in Figure 3 we give the quotients $MSE(\widehat{q}_\alpha(t))/MSE(\widehat{q}^*_\alpha(t))$ for the case $n = 100$ along the nine deciles, for $\alpha = 0.25, 0.5, 0.75$ and three different censoring levels: 0%, 33%, and 67% (from left to right panel). In this Figure 3 we see that the relative benefits associated to the monotone estimator are greater in the heavily censored situation; indeed, the relative deficiency of the unrestricted estimator may be above 300% when the percentage of

Table 2: Bias and MSE of $\hat{q}_{0.5}$ and $\hat{q}^*_{0.5}$ for $n = 250$, and for different levels of censoring and deciles

		BIAS		MSE	
		$\hat{q}$	$\hat{q}^*$	$\hat{q}$	$\hat{q}^*$
$CP = 0\%$	D_1	-0.00107	-0.00330	0.00134	0.00134
	D_2	-0.00088	-0.00486	0.00134	0.00133
	D_5	-0.00082	-0.01130	0.00146	0.00143
	D_8	-0.00118	-0.02866	0.00214	0.00215
	D_9	-0.00069	-0.04504	0.00338	0.00352
$CP = 15\%$	D_1	0.00055	-0.00211	0.00159	0.00157
	D_2	0.00072	-0.00380	0.00157	0.00154
	D_5	0.00181	-0.01073	0.00174	0.00161
	D_8	0.00323	-0.03070	0.00295	0.00259
	D_9	0.00281	-0.05037	0.00495	0.00439
$CP = 33\%$	D_1	0.00087	-0.00228	0.00194	0.00190
	D_2	0.00122	-0.00448	0.00200	0.00194
	D_5	0.00156	-0.01457	0.00236	0.00215
	D_8	0.00563	-0.03941	0.00455	0.00372
	D_9	0.01079	-0.06364	0.00983	0.00664
$CP = 67\%$	D_1	0.00358	-0.00357	0.00409	0.00379
	D_2	0.00397	-0.00827	0.00450	0.00401
	D_5	0.01011	-0.02763	0.00816	0.00539
	D_8	0.01699	-0.07901	0.02426	0.01217
	D_9	-0.02322	-0.12661	0.03162	0.02314

censoring is about 67%. This agrees with the variance reduction which is achieved by using the restricted estimator. Finally, we can not deduce any systematic influence of the α parameter nor the decile in the relative performance of both estimators. However, Figure 3 suggests that for small α the relative efficiency of the monotone estimator increases at large deciles when there is some censoring, while for moderate or large values of α the maximum relative efficiency may be reached at intermediate deciles depending on the censoring degree. This trade-off among α, the censoring level, and the deciles of X is also evident from the asymptotic variance of $\widehat{q}_\alpha(t)$, see e.g. Theorem 6.1 in Chung (1989).

4 Real data illustration

For illustration purposes, we consider the PBC data set reported and widely explained in Fleming and Harrington (1991), with n = 312 individuals. In this example, the variable X denotes survival time (in days) for primary biliary cirrhosis (PBC) patients. Censoring from the right is provoked by the end of following-up or by liver transplantation (187 censored times or about 60% of censoring). It is known that the survival prognosis is grealty influenced by the level of edema, so we consider three different groups of patients according to this variable. The first group (edema=0) corresponds to patients with no edema; patients in second group (edema=0.5) had an untreated or a successfully treated edema; while the third group (edema=1) corresponds to patients with an unsuccessfully

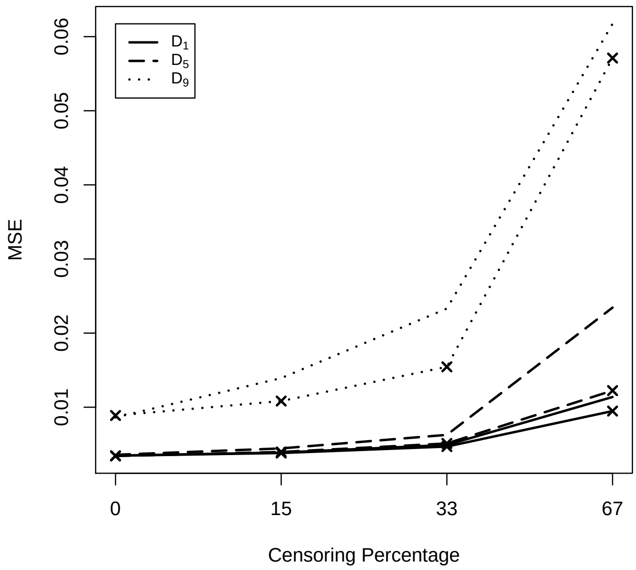

Figure 2: MSEs of $\hat{q}_{0.5}$ and $\hat{q}^*_{0.5}$ (crosses) for three different deciles and different censoring degrees

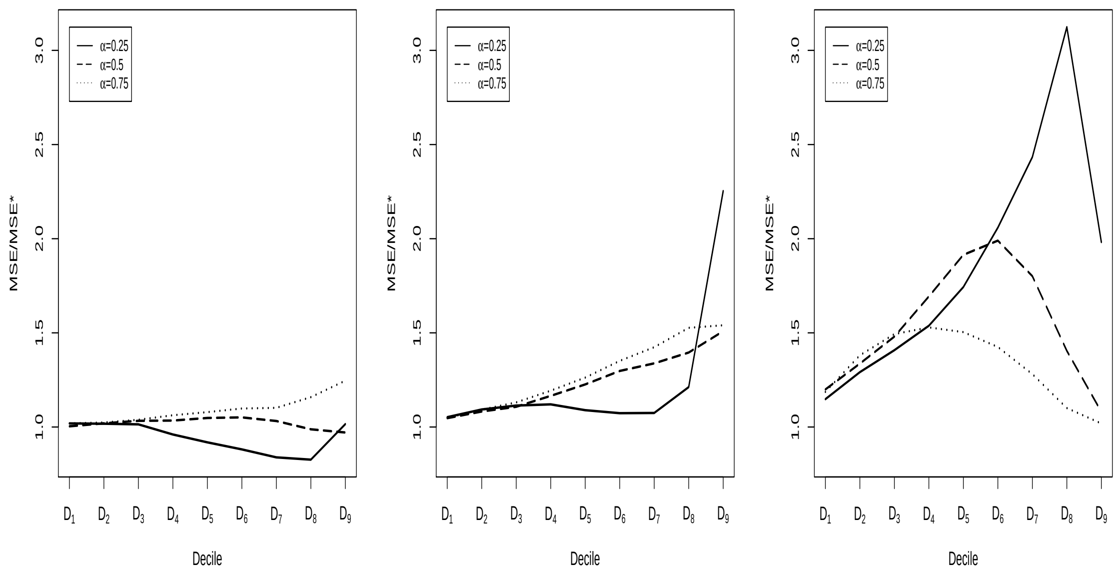

Figure 3: $MSE(\widehat{q}_\alpha(t))/MSE(\widehat{q}^*_\alpha(t))$ along the nine deciles, for different α's and censoring levels: 0% (left), 33% (center), and 67% (right)

Table 3: Number of cases and deaths in each group, and median survival (in days)

Level of Edema	Number of cases	Deaths	Median Survival
0	263	89	3584
0.5	29	17	1576
1	20	19	299

treated edema. In Table 3 we report the number of cases and deaths in each group, together with the median survival. From this Table we see that an increasing value of edema is associated to a poorer survival prognosis.

In Figure 4 we give the 25%-percentile residual life function for the three groups of edema, when estimated by using the restricted or the unrestricted estimators. For the first group (edema=0), the unrestricted estimator suggests a decreasing shape; this is not surprising, since the cumulative hazard plot for this group (see Figure 5) reveals an increasing hazard rate, which is a characteristic property of the decreasing percentile residual life populations (e.g. Joe and Proschan, 1984b). In this case, by using the monotone estimator we get some smoothing of the curve which results in a nicer estimator. The other two groups offer a different situation, since the unrestricted estimator is not supporting in principle the monotonicity of the percentile residual life function. This could be explained by the existence of a non-increasing hazard rate for the last two groups; indeed, the corresponding Nelson-Aalen estimators depict a concave form (Figure 5).

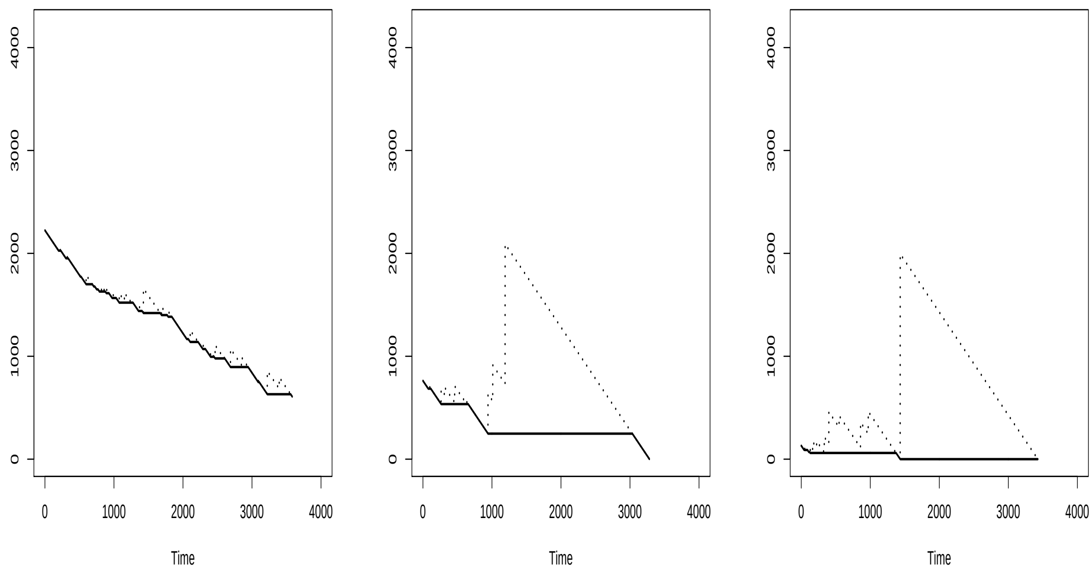

Figure 4: Illustration of $\hat{q}_{0.25}$ (dotted) and $\hat{q}^*_{0.25}$ (solid) for the three groups of edema: 0 (left), 0.5 (center), and 1 (right)

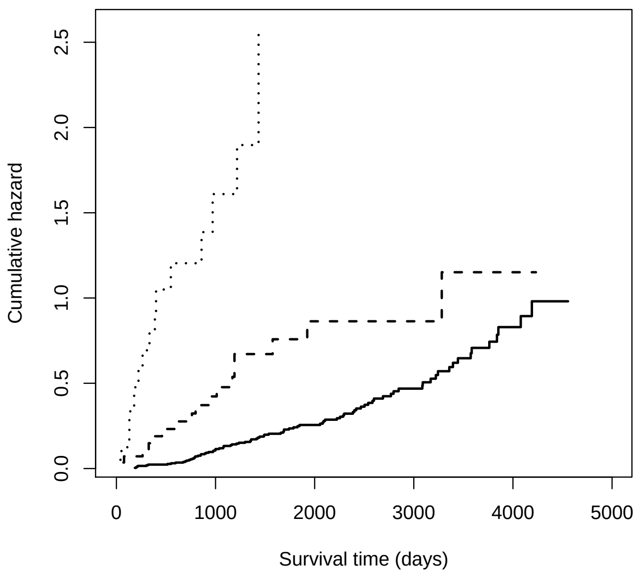

Figure 5: Nelson-Aalen estimators of the cumulative hazard for the three groups of edema: 0 (solid), 0.5 (dashed), and 1 (dotted)

5 Main conclusions

In this paper a new estimator for the percentile residual life function under random censorship has been introduced. The new estimator is suitable when the percentile residual life function is monotone decreasing. A law of the iterated logarithm has been established; besides, it has been demonstrated that the monotone estimator is $\sqrt{(n)}$-equivalent to the unrestricted one. The finite sample performance of the new estimator has been investigated through simulations. In particular, it has been illustrated that much efficiency may be gained through the using of the monotone estimator when the sample size is low and the censoring level is high. A real data illustration has been provided.

A key question in practice is whether one should assume beforehand that the percentile residual life function is monotone. Our real data application has shown that this is not always the case. It would be very interesting to develop goodness-of-fit tests for the monotonicity assumption. A possible way of doing that is through a proper distance between the restricted and the unrestricted estimators. This topic is currently under research. Finally, the application of the ideas in this paper to estimate monotone increasing residual life functions is possible, and completely analogous estimators are obtained in such a case.

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