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Report 08/07

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The Owen value in minimum cost spanning tree problems with coalition structure*

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Abstract

Minimum cost spanning tree problems with coalition structure were introduced in Bergantiños and Gómez-Rúa (2007). Moreover, a rule for dividing the cost of connecting all the agents to the source in this kind of problems is defined. In this paper we prove that this rule coincides with the Owen value of the TU game associated with the irreducible matrix.

Keywords: minimum cost spanning tree problems with coalition structure. TU game, Owen value.

1 Introduction

The classical minimum cost spanning tree problems ($mcstp$) model situations where a group of agents (denoted by N), located at different geographical places, want a particular service which can only be provided by a common supplier, called the source (denoted by 0). Agents will be served through connections which involve some cost. Moreover, they do not care whether they are connected directly or indirectly to the source. This situation is described by a symmetric matrix $C = (c_{ij})_{i,j \in N \cup \{0\}}$, where c_{ij} denotes the

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connection costs between i and j $(i, j \in N \cup \{0\})$. There are many economic situations that can be modeled in this way. For instance, several towns may draw power from a common power plant, and hence have to share the cost of the distribution network (Dutta and Kar, 2004). Bergantiños and Lorenzo (2004, 2005) studied a real situation where villagers had to pay the cost of constructing pipes from their respective houses to a water supplier. Other examples include communication networks, such as telephone, internet, or cable television.

We assume that the agents construct a minimum cost spanning tree (mt). The question is how to divide the cost associated with the mt between the agents. Different rules give different answers to this question.

There are several rules studied in the literature. For instance, the rules studied in Bird (1976), Kar (2002), and Dutta and Kar (2004). Feltkamp *et al.* (1994) defined a rule called Equal Remaining Obligations rule (ERO). ERO is called the *P-value* in Branzei *et al.* (2004).

One of the most important topics is the axiomatic characterization of rules. The idea is to propose desirable properties and to find out which of them characterize each rule. Properties often help agents to compare different rules and to decide which rule is preferred in a particular situation.

However, this model ignores the fact that some group of agents are located in the same city or village. In Bergantiños and Gómez-Rúa (2007) the minimum cost spanning tree problems with coalition structure are introduced. In these problems we include this fact in the model. We do it by considering an extra element: a partition $G = \{G^1, ..., G^m\}$ of the set of agents. For each $k = 1, ..., m$, G^k represents the coalition of agents located in the same village or city.

In Bergantiños and Gómez-Rúa (2007) we introduced a rule for this kind of problems, F and we provided an axiomatic characterization of this rule. The proposed rule is a generalization of the rule provided by Bergantiños and Vidal-Puga (2007a), which we denote as φ.

Owen (1977) introduced a value for transferable utility (TU, for short) games with a coalition structure. It is assumed that the agents are partitioned into different coalitions. Moreover, our objective is to divide the value of the grand coalition among the agents taking into account the coalition structure. Owen (1977) proved that his value generalizes the Shapley value.

Bird (1976) defined the minimal network and the TU game (N, v_C) associated with an *mcstp* (N_0, C). Bergantiños and Vidal-Puga (2007a) defined the irreducible matrix C^* associated with an *mcstp* (N_0, C) through the minimal network. The rule φ in *mcstp* is defined as the Shapley value of the *TU* game (N, v_{C^*}). As we pointed before, the rule F in *mcstp* with coalition structure generalizes the rule φ. We may ask if there exists any relationship between F and the Owen value of (N, v_{C^*}, G). The answer is not trivial because (N, v_{C^*}, G) does not appear in the definition of F. Nevertheless, in this paper we prove that F coincides with the Owen value of (N, v_{C^*}, G).

The paper is organized as follows. In Section 2 we introduce the model. In Section 3 we prove that F coincides with the Owen value.

2 The model

Let $\mathcal{N} = \{1, 2, ...\}$ be the set of all possible agents. Given a finite set $N \subset \mathcal{N}$, let Π_N be the set of all permutations over N. Given $\pi \in \Pi_N$, let $Pre\,(i, \pi)$ denote the set of the elements of N which come before i in the order given by π, *i.e.* $Pre\,(i, \pi) = \{j \in N : \pi\,(j) < \pi\,(i)\}$. Given $S \subset N$, let π_S denote the order induced by π among the agents in S.

We are interested in networks whose nodes are elements of a set $N_0 = N \cup \{0\}$, where $N \subset \mathcal{N}$ is finite and 0 is a special node called the *source*. Usually we take $N = \{1, ..., n\}$.

A *cost matrix* $C = (c_{ij})_{i,j \in N_0}$ on N represents the cost of direct link between any pair of nodes. We assume that $c_{ij} = c_{ji} \geq 0$ for each $i, j \in N_0$ and $c_{ii} = 0$ for each $i \in N_0$. Since $c_{ij} = c_{ji}$ we work with undirected arcs, *i.e.* $(i, j) = (j, i)$.

We denote the set of all cost matrices over N as $\mathcal{C}^N$. Given C, $C' \in \mathcal{C}^N$ we say $C \leq C'$ if $c_{ij} \leq c'_{ij}$ for all $i, j \in N_0$.

A *minimum cost spanning tree problem*, briefly an *mcstp*, is a pair (N_0, C) where $N \subset \mathcal{N}$ is a finite set of agents, 0 is the source, and $C \in \mathcal{C}^N$ is the cost matrix.

Given an *mcstp* (N_0, C), we define the *mcstp* induced by C in $S \subset N$ as (S_0, C).

A *network* g over N_0 is a subset of $\{(i, j) : i, j \in N_0\}$. The elements of g are called *arcs*. Given a network g over N_0 and $S \subset N_0$ we denote by g_S the network induced by g among the elements of S. Namely, $g_S = \{(i, j) \in g : \{i, j\} \subset S\}$.

Given a network g and a pair of nodes i and j, a *path* from i to j in g is a sequence of different arcs $\{(i_{h-1}, i_h)\}_{h=1}^{l}$ satisfying $(i_{h-1}, i_h) \in g$ for all $h \in \{1, 2, ..., l\}$, $i = i_0$, and $j = i_l$.

A *tree* is a network such that for all $i \in N$ there is a unique path from i to the source. If t is a tree, we usually write $t = \{(i^0, i)\}_{i \in N}$ where i^0 represents the first agent in the unique path in t from i to 0.

Let $\mathcal{G}^N$ denote the set of all networks over N_0. Let $\mathcal{G}_0^N$ denote the set of all networks where every agent $i \in N$ is connected to the source, *i.e.* there exists a path from i to 0 in the network.

Given an *mcstp* (N_0, C) and $g \in \mathcal{G}^N$, we define the *cost* associated with g as

$$c(N_0, C, g) = \sum_{(i,j) \in g} c_{ij}.$$

When there is no ambiguity, we write $c(g)$ or $c(C, g)$ instead of $c(N_0, C, g)$.

A *minimum cost spanning tree* for (N_0, C), briefly an *mt*, is a tree t over N_0 such that $c(t) = \min_{g \in \mathcal{G}_0^N} c(g)$. It is well-known that an *mt* exists, even though it is not necessarily unique. Given an *mcstp* (N_0, C), we denote the cost associated with any *mt* as $m(N_0, C)$.

There are several algorithms in the literature to construct an mt. Prim (1957) provides such an algorithm. The idea of this algorithm is as follows: sequentially, the agents connect to the source. At each stage, the cheapest arc between the connected and the unconnected agents is added. This algorithm leads to a tree, but this is not always unique.

Given an *mcstp* (N_0, C) and an *mt* t, Bird (1976) defined the *minimal network* (N_0, C^t) associated with t as follows: $c_{ij}^t = \max_{(k,l) \in g_{ij}} \{c_{kl}\}$, where g_{ij} denotes the unique path in t from i to j. Even though g_{ij} depends on the choice of t, c_{ij}^t is independent of the chosen t. Proof of this can be found in Aarts and Driessen (1993).

The *irreducible form* of an *mcstp* (N_0, C) is defined as the minimal network (N_0, C^*) associated with a particular *mt* t. If (N_0, C^*) is an *irreducible form*, we say that C^* is an *irreducible matrix*.

A (*cost allocation*) *rule* is a function f such that $f(N_0, C) \in \mathbb{R}^N$ and $\sum_{i \in N} f_i(N_0, C) = m(N_0, C)$ for each *mcstp* (N_0, C). As usual, $\psi_i(N_0, C)$ represents the cost allocated to agent i.

Notice that we implicitly assume that the agents build an *mt*. As far as we know, all the rules proposed in the literature make this assumption.

A *coalitional game with transferable utility*, briefly a *TU game*, is a pair

(N,v) where $v : 2^N \to \mathbb{R}$ satisfies $v(\varnothing) = 0$. $Sh(N,v)$ denotes the Shapley value (Shapley (1953)) of (N,v).

For each *mcstp* (N_0, C), Bird (1976) introduces the *TU* game (N, v_C). For each coalition $S \subset N$,

$$v_C(S) = m(S_0, C).$$

On the other hand, Bergantiños and Vidal-Puga (2007a) defined the rule φ as

$$\varphi(N_0, C) = Sh(N, v_{C^*})$$

where C^* is the irreducible matrix associated with C. Bergantiños and Vidal-Puga (2007e) proved that, surprisingly, φ coincides with ERO. This rule is also studied in Bergantiños and Vidal-Puga (2007b, 2007c, 2007d).

In Bergantiños and Gómez-Rúa (2007) we introduced the *mcstp* with coalition structure. An *mcstp with coalition structure* is a triple (N_0, C, G) where (N_0, C) is a *mcstp*, $G = \{G^1, ..., G^m\}$ is a partition of N and for each $k = 1, ...m$

$$\max_{i,j \in G^k} \{c_{ij}\} \leq \min_{i \in G^k, j \notin G^k} \{c_{ij}\}.$$

A *rule* in *mcstp* with coalition structure is a function f such that $f(N_0, C, G) \in \mathbb{R}^N$ and $\sum_{i \in N} f_i(N_0, C, G) = m(N_0, C)$ for each *mcstp* (N_0, C).

As in classical *mcstp*, the main objective is to divide the cost associated with an *mt* among the agents in a fair way.

A rule for *mcstp* with coalition structure, F is defined in Bergantiños and Gómez-Rúa (2007). The intuitive idea of this rule is as follows. F can be considered as a "two-steps" rule. In the first step we compute the amount that each coalition should pay in order to be connected to the source. We do it applying the rule φ defined in Bergantiños and Vidal-Puga (2007a).

In the second step we decide the amount that each agent of each coalition has to pay. For each coalition G^k, we consider the *mcstp* inside each coalition (G_0^k, C^φ). In this *mcstp*, the connection cost between two agents of G^k is the same as in C but the connection cost between any agent of G^k and the source is the amount assigned to the coalition G^k in the first step.

Formally this rule is defined as follows: Given the *mcstp* with coalition structure (N_0, C, G), with $G = \{G^1, ..., G^m\}$ and $M = \{1, ..., m\}$, we define the *mcstp among coalitions*, (M_0, C^G) as follows:

- $M_0 = \{0, 1, ..., m\}$.
- C^G is the cost matrix and for each $k, k' \in M_0$, the connection cost between k and k' is denoted by

$$c^G_{kk'} = \min_{i \in G^k, j \in G^{k'}} \{c_{ij}\}.$$

Let (N_0, C, G) be an *mcstp* with coalition structure and $i \in G^k$. We define

$$F_i(N_0, C, G) = \varphi_i\left(G^k_0, C^\varphi\right)$$

where $C^\varphi = (c^\varphi_{jj'})_{j,j' \in G^k_0}$ is defined as

$$c^\varphi_{jj'} = \begin{cases} c_{jj'} & \text{if } 0 \notin \{j, j'\} \\ \varphi_k\left(M_0, C^G\right) & \text{if } 0 \in \{j, j'\}. \end{cases}$$

Before introducing the result of the paper, we present two Lemmas which will be used often in the proofs of the main result.

Lemma 1 *(Bergantiños and Vidal-Puga (2007a)) If $C^* = (c^*_{ij})_{i,j \in N_0}$ is an irreducible matrix, then for all $S \subset N_0$, $i \notin S$ we have that*

$$v_{C^*}(S \cup \{i\}) - v_{C^*}(S) = \min_{j \in S_0} \left\{c^*_{ij}\right\}.$$

Lemma 2 *(Bergantiños and Gómez-Rúa (2007)) Given (N_0, C, G) we can find an mt t in (N_0, C) satisfying:*

(i) For each $k = 1, ..., m$, t_{G^k} induces an *mt* in $\left(G^k, C\right)$.
(ii) $\left\{(k, k') : \exists i \in G^k, j \in G^{k'} \text{ with } (i, j) \in t\right\}$ is an *mt* in $\left(M_0, C^G\right)$.
(iii) For each $k = 1, ..., m$ and each $i \in G^k$, $t_{G_k} \cup \{(0, i)\}$ is an *mt* in (G^k_0, C^φ).

3 An approach using TU games

Owen (1977) introduces a value for TU games with coalition structure. It is assumed that agents are partitioned into different coalitions. Moreover, the objective is to divide the value of the grand coalition among the agents taking into account the coalition structure. Owen (1977) proves that his value generalizes the Shapley value.

The rule φ in *mcstp* is defined as the Shapley value of the TU game (N, v_{C^*}). The rule F in *mcstp* with coalition structure generalizes the rule φ. We can ask if there is some relationship between F and the Owen value of (N, v_{C^*}, G). The answer is not trivial because (N, v_{C^*}, G) does not appear in the definition of F. Nevertheless, we will prove that F coincides with the Owen value of (N, v_{C^*}, G).

We first introduce the Owen value formally. A *TU game with coalition structure* is a triple (N, v, G) where (N, v) is a TU game and $G = \{G^1, ..., G^m\}$ is a partition of N.

We say that a permutation $\pi \in \Pi_N$ is *admissible* with respect to G if given $i, i' \in G^k \in G$ and $j \in N$ with $\pi(i) < \pi(j) < \pi(i')$, then $j \in G^k$. We denote by Π^G the set of all permutations over N admissible with respect to G.

Given (N, v, G) and $i \in G^k \in G$, the Owen value is defined as

$$Ow_i(N, v, G) = \frac{1}{|\Pi^G|} \sum_{\pi \in \Pi^G} \left[v\left(Pre(i, \pi) \cup \{i\}\right) - v\left(Pre(i, \pi)\right)\right].$$

Now we present our result.

Theorem 3 *For each mcstp with coalition structure* (N_0, C, G) *and* $i \in G^k \in G$,

$$F_i(N_0, C, G) = Ow_i(N, v_{C^*}, G).$$

Proof. *Let* (N_0, C, G) *be an mcstp with coalition structure and* $i \in G^k \in G$. *We will prove the result in several Claims.*

For each mcstp with coalition structure (N_0, C, G) *and each* $G^k \in G$, *let* (N_0', C', G') *the problem obtained from* (N_0, C, G) *by considering that the rest of the coalitions have a unique agent whose connection cost to the rest of the agents is given by* $\left(M_0, C^G\right)$. *Namely,* $N' = G^k \cup \left(\bigcup_{l \neq k} \{i_l\}\right)$, $G' =$

$\{G^k\} \cup \{\{i_l\}\}_{l\in M\backslash\{k\}}$, *and* C' *is defined as follows: if* $i, j \in G^k \cup \{0\}$, *then* $c'_{ij} = c_{ij}$. *If* $i \in G^k$ *and* $j = i_l$ *with* $l \in M\backslash\{k\}$, *then* $c'_{ij} = c^G_{kl}$. *If* $i = 0$ *and* $j = i_l$ *with* $l \in M\backslash\{k\}$, *then* $c'_{ij} = c^G_{0l}$. *If* $i = i_l$, $j = i_{l'}$ *and* $k \notin \{l, l'\}$, *then* $c'_{ij} = c^G_{ll'}$.

In Bergantiños and Gómez-Rúa (2007) (Claim 13 in Proposition 11) we prove that:

$$F_i(N_0, C, G) = F_i(N'_0, C', G').$$

We proceed with several claims.

Claim 4 $Ow_i(N, v_{C^*}, G) = Ow_i(N', v_{C'^*}, G')$.

Proof. *We know that*

$$\begin{aligned} Ow_i(N, v_{C^*}, G) &= \frac{1}{|\Pi^G|}\sum_{\pi\in\Pi^G}\left[v_{C^*}(Pre(i,\pi)\cup\{i\}) - v_{C^*}(Pre(i,\pi))\right] \text{ and} \\ Ow_i(N', v_{C'^*}, G') &= \frac{1}{|\Pi^{G'}|}\sum_{\pi'\in\Pi^{G'}}\left[v_{C'^*}(Pre(i,\pi')\cup\{i\}) - v_{C'^*}(Pre(i,\pi'))\right]. \end{aligned}$$

For each $\pi' \in \Pi^{G'}$ *let* $O(\pi')$ *denote the set of orders of* Π^G *inducing the same order than* π' *among the agents in* G^k *and among the coalitions. Namely,* $O(\pi')$ *is the set of orders* $\pi \in \Pi^G$ *satisfying two conditions:*

1. $\pi_{G^k} = \pi'_{G^k}$.

2. Given $j \in G^l$, $j' \in G^{l'}$, $k \notin \{l, l'\}$ *we have that* $\pi(j) < \pi(j')$ *if and only if* $\pi'(i_l) < \pi'(i_{l'})$.

Thus, for all $\pi' \in \Pi^{G'}$, $|O(\pi')| = \prod_{l\neq k}(|G^l|!)$.

We now prove that given $\pi' \in \Pi^{G'}$ *and* $\pi \in O(\pi')$, *we have that*

$$v_{C'^*}(Pre(i,\pi')\cup\{i\}) - v_{C'^*}(Pre(i,\pi')) = v_{C^*}(Pre(i,\pi)\cup\{i\}) - v_{C^*}(Pre(i,\pi)).$$

By Lemma 1,

$$\begin{aligned} v_{C'^*}(Pre(i,\pi')\cup\{i\}) - v_{C'^*}(Pre(i,\pi')) &= \min_{j\in Pre(i,\pi')_0}\left\{c'^*_{ij}\right\} \text{ and} \\ v_{C^*}(Pre(i,\pi)\cup\{i\}) - v_{C^*}(Pre(i,\pi)) &= \min_{j\in Pre(i,\pi)_0}\left\{c^*_{ij}\right\}. \end{aligned}$$

We consider two cases:

- $Pre\,(i,\pi') \cap G^k \neq \varnothing$.

We know that for all $l \in M$, $\max_{j,j' \in G^l} \{c_{jj'}\} \leq \min_{j \in G^l, j' \notin G^l} \{c_{jj'}\}$. Because of the definition of the irreducible matrix as the minimal network associated with the minimal tree given by Lemma 2, it is easy to deduce that for all $l \in M$, $\max_{j,j' \in G^l} \left\{c^*_{jj'}\right\} \leq \min_{j \in G^l, j' \notin G^l} \left\{c^*_{jj'}\right\}$. Now,

$$v_{C^*}\left(Pre\,(i,\pi) \cup \{i\}\right) - v_{C^*}\left(Pre\,(i,\pi)\right) = \min_{j \in \left(Pre(i,\pi) \cap G^k\right)_0} \left\{c^*_{ij}\right\}.$$

Analogously,

$$v_{C'^*}\left(Pre\,(i,\pi') \cup \{i\}\right) - v_{C'^*}\left(Pre\,(i,\pi')\right) = \min_{j \in \left(Pre(i,\pi') \cap G^k\right)_0} \left\{c'^*_{ij}\right\}.$$

Because of the definition of C', $c^*_{ij} = c'^*_{ij}$ for all $j \in G^k$. Since $\pi \in O\,(\pi')$, $Pre\,(i,\pi) \cap G^k = Pre\,(i,\pi') \cap G^k$.

Then, the result holds.

- $Pre\,(i,\pi') \cap G^k = \varnothing$.

Let t be the mt given by Lemma 2. We can compute C^* as the minimal network associated with t.

By Lemma 2, we know that t_{G^k} is an mt in $\left(G^k, C\right)$ and the tree $t^G = \left\{(k,k') : \exists i \in G^k, j \in G^{k'} \text{ with } (i,j) \in t\right\}$ is an mt in $\left(M_0, C^G\right)$. Now it is easy to deduce that $t' = t_{G^k} \cup t^G$ induces an mt in (N'_0, C', G'). Then, we can compute C'^* as the minimal network associated with t'.

Since $Pre\,(i,\pi') \cap G^k = \varnothing$, we can assume that $\min_{j \in Pre(i,\pi')_0} \left\{c'^*_{ij}\right\} = c'^*_{ii_l}$ with $l \neq k$ ($i_l = i_0 = 0$ is also possible). Let g_{ii_l} be the unique path in t' joining i and i_l. Then, $c'^*_{ii_l} = c'_{i_a i_b}$ where $(i_a, i_b) \in g_{ii_l}$. By definition of C', $c'_{i_a i_b} = c_{j_a j_b}$ where $j_a \in G^a$ and $j_b \in G^b$.

Since $\min_{j \in Pre(i,\pi')_0} \left\{c'^*_{ij}\right\} = c'^*_{ii_l}$, $G^l \subset Pre\,(i,\pi)_0$. Now, there exists $j_l \in G^l$ such that

$$\min_{j \in Pre(i,\pi)_0} \left\{c^*_{ij}\right\} = c^*_{ij_l}.$$

Because of the definition of C^* as the minimal network associated with t we have that (j_a, j_b) belongs to the unique path in t joining i and j_l. Thus,

$$\min_{j \in Pre(i,\pi)_0} \left\{c^*_{ij}\right\} = c^*_{ij_l} \geq c_{j_a j_b} = \min_{j \in Pre(i,\pi')_0} \left\{c'^*_{ij}\right\}.$$

Using arguments similar to those used above we can prove that $\min_{j \in Pre(i,\pi)_0} \left\{ c^*_{ij} \right\} \leq \min_{j \in Pre(i,\pi')_0} \left\{ c'^*_{ij} \right\}$.

It is easy to see that, $\left|\Pi^G\right| = m! \left(\prod_{l=1}^{m} \left(\left|G^l\right|! \right) \right)$, $\left|\Pi^{G'}\right| = m! \left(\left|G^k\right|! \right)$, *and for each* $\pi' \in \Pi^{G'}$, $|O(\pi')| = \prod_{l \in M \setminus \{k\}} \left(\left|G^l\right|! \right)$. *Thus,*

$$\begin{aligned} Ow_i(N, v_{C^*}, G) &= \frac{1}{|\Pi^G|} \sum_{\pi' \in \Pi^{G'}} \sum_{\pi \in O(\pi')} \left[v_{C^*}(Pre(i,\pi) \cup \{i\}) - v_{C^*}(Pre(i,\pi)) \right] \\ &= \frac{1}{|\Pi^G|} \sum_{\pi' \in \Pi^{G'}} \sum_{\pi \in O(\pi')} \left[v_{C'^*}(Pre(i,\pi') \cup \{i\}) - v_{C'^*}(Pre(i,\pi')) \right] \\ &= \frac{1}{|\Pi^G|} \sum_{\pi' \in \Pi^{G'}} \left(\prod_{l \in M \setminus \{k\}} \left|G^l\right|! \right) \left[v_{C'^*}(Pre(i,\pi') \cup \{i\}) - v_{C'^*}(Pre(i,\pi')) \right] \\ &= \frac{1}{|\Pi^{G'}|} \sum_{\pi' \in \Pi^{G'}} \left[v_{C'^*}(Pre(i,\pi') \cup \{i\}) - v_{C'^*}(Pre(i,\pi')) \right] \\ &= Ow_i(N', v_{C'^*}, G'). \end{aligned}$$

■

Thus, we can assume that (N_0, C, G) *satisfies that* $\left|G^l\right| = 1$ *for all* $l \neq k$.

Claim 5 *Let* $\pi \in \Pi^G$ *such that* $Pre(i,\pi) \cap G^k \neq \varnothing$. *Thus,*

$$v_{C^*}(Pre(i,\pi) \cup \{i\}) - v_{C^*}(Pre(i,\pi)) = v_{(C^\varphi)^*}(Pre(i,\pi_{G^k}) \cup \{i\}) - v_{(C^\varphi)^*}(Pre(i,\pi_{G^k})).$$

Proof. *We have seen in the proof of Claim 4 that*

$$v_{C^*}(Pre(i,\pi) \cup \{i\}) - v_{C^*}(Pre(i,\pi)) = \min_{j \in \left(Pre(i,\pi) \cap G^k\right)_0} \left\{ c^*_{ij} \right\}.$$

By Lemma 1,

$$v_{(C^\varphi)^*}(Pre(i,\pi_{G^k}) \cup \{i\}) - v_{(C^\varphi)^*}(Pre(i,\pi_{G^k})) = \min_{j \in Pre\left(i,\pi_{G^k}\right)_0} \left\{ c^{\varphi *}_{ij} \right\}.$$

*Let t be an mt as in Lemma 2. Thus, $t^k = t_{G^k} \cup \{(0,i)\}$ is an mt in $\left(G_0^k, C^\varphi\right)$. Because of the proof of Lemma 2, (See Bergantiños and Gómez-Rúa (2007), Lemma 3)***,** for all $(j,j') \in t^k$, $c^\varphi_{jj'} \leq c^\varphi_{0i}$. Since $(C^\varphi)^*$ is the minimal network associated with t^k, we deduce that*

$$v_{(C^\varphi)^*}\left(Pre\left(i,\pi_{G^k}\right)\cup\{i\}\right) - v_{(C^\varphi)^*}\left(Pre\left(i,\pi_{G^k}\right)\right) = \min_{j\in\left(Pre\left(i,\pi_{G^k}\right)\cap G^k\right)_0}\left\{c^{\varphi*}_{ij}\right\}.$$

Since $Pre\left(i,\pi_{G^k}\right)\cap G^k = Pre\left(i,\pi\right)\cap G^k$, it is enough to prove that for all $j \in Pre\left(i,\pi\right)\cap G^k$, $c^{\varphi*}_{ij} = c^*_{ij}$. Let $j \in G^k$.

We know that $(C^\varphi)^*$ is the minimal network associated with t^k and C^* is the minimal network associated with t. Let g^φ_{ij} denote the unique path in t^k joining i and j. Let g_{ij} denote the unique path in t joining i and j. By Lemma 2, t_{G^k} is a tree in $\left(G^k, C^\varphi\right)$. Since $t^k_{G^k} = t_{G^k}$, we deduce that $g^\varphi_{ij} = g_{ij} \subset t_{G^k}$. Then,

$$c^{\varphi*}_{ij} = \max_{(a,b)\in g^\varphi_{ij}}\left\{c^\varphi_{ab}\right\} = \max_{(a,b)\in g_{ij}}\left\{c^\varphi_{ab}\right\}.$$

By definition of C^φ, $c^\varphi_{ij'} = c_{ij'}$ for all $j' \in G^k$. Now, $c^{\varphi*}_{ij} = \max_{(a,b)\in g_{ij}}\{c_{ab}\} = c^*_{ij}$. ■

Claim 6 *Let $\pi \in \Pi^G$ such that $Pre\left(i,\pi\right)\cap G^k = \varnothing$. Let π' denote the order induced by π in M. Namely, $\pi'(l) < \pi'(l')$ if and only if there exist $j \in G^l$ and $j' \in G^{l'}$ such that $\pi(j) \leq \pi(j')$. Since $\pi \in \Pi^G$, π' is well defined. Thus,*

1. *$v_{C^*}\left(Pre\left(i,\pi\right)\cup\{i\}\right) - v_{C^*}\left(Pre\left(i,\pi\right)\right) = v_{(C^G)^*}\left(Pre\left(k,\pi'\right)\cup\{k\}\right) - v_{(C^G)^*}\left(Pre\left(k,\pi'\right)\right)$.*

2. *$v_{(C^\varphi)^*}\left(Pre\left(i,\pi_{G^k}\right)\cup\{i\}\right) - v_{(C^\varphi)^*}\left(Pre\left(i,\pi_{G^k}\right)\right) = \varphi_k\left(M_0, C^G\right)$.*

Proof. *1. By Lemma 1,*

$$\begin{aligned} v_{C^*}\left(Pre\left(i,\pi\right)\cup\{i\}\right) - v_{C^*}\left(Pre\left(i,\pi\right)\right) &= \min_{l\in Pre(i,\pi)_0}\left\{c^*_{il}\right\} \text{ and} \\ v_{(C^G)^*}\left(Pre\left(k,\pi'\right)\cup\{k\}\right) - v_{(C^G)^*}\left(Pre\left(k,\pi'\right)\right) &= \min_{l\in Pre(k,\pi')_0}\left\{c^{G*}_{kl}\right\}. \end{aligned}$$

It is obvious that $Pre\left(i,\pi\right)$ coincides with $Pre\left(k,\pi'\right)$. Let t be an mt as in Lemma 2. Thus, $t^G = \left\{(k,k') : \exists i \in G^k, j \in G^{k'} \text{ with } (i,j)\in t\right\}$ is an mt in $\left(M_0, C^G\right)$. Using arguments similar to those used in the proof of

Claim 5, we can prove that for all $l \in Pre(i,\pi)$ and its equivalent coalition $l' \in Pre(k,\pi')$, $c^*_{il} = c^{G*}_{kl'}$.

2. By Lemma 1,

$$v_{(C^\varphi)^*}\left(Pre\left(i,\pi_{G^k}\right)\cup\{i\}\right) - v_{(C^\varphi)^*}\left(Pre\left(i,\pi_{G^k}\right)\right) = \min_{j\in Pre\left(i,\pi_{G^k}\right)_0}\left\{c^{\varphi*}_{ij}\right\}.$$

Since $Pre(i,\pi)\cap G^k = \varnothing$, $Pre\left(i,\pi_{G^k}\right)_0 = \{0\}$. Thus,

$$v_{(C^\varphi)^*}\left(Pre\left(i,\pi_{G^k}\right)\cup\{i\}\right) - v_{(C^\varphi)^*}\left(Pre\left(i,\pi_{G^k}\right)\right) = c^{\varphi*}_{0i}.$$

By Lemma 2, (iii), $t^k = t_{G^k}\cup\{(0,i)\}$ is an mt in $\left(G^k_0, C^\varphi\right)$. Since $C^{\varphi*}$ is the minimal network associated with t^k,

$$c^{\varphi*}_{0i} = c^{\varphi}_{0i} = \varphi_k\left(M_0, C^G\right).$$

■

Claim 7 $F_i(N_0,C,G) = Ow_i(N, v_{C^*}, G)$.
Proof. *We know that*

$$\begin{aligned}
F_i(N_0,C,G) &= \varphi_i\left(G^k_0, C^\varphi\right) = Sh_i\left(G^k, v_{(C^\varphi)^*}\right)\\
&= \frac{1}{|\Pi_{G^k}|}\sum_{\pi\in\Pi_{G^k}}\left[v_{(C^\varphi)^*}\left(Pre(i,\pi)\cup\{i\}\right) - v_{(C^\varphi)^*}\left(Pre(i,\pi)\right)\right].
\end{aligned}$$

Let X^k_1, X^k_2 the partition of Π_{G^k} where

$$\begin{aligned}
X^k_1 &= \left\{\pi\in\Pi_{G^k} : Pre(i,\pi)\cap G^k \neq \varnothing\right\} \text{ and}\\
X^k_2 &= \left\{\pi\in\Pi_{G^k} : Pre(i,\pi)\cap G^k = \varnothing\right\}.
\end{aligned}$$

Since $|\Pi_{G^k}| = \left|G^k\right|!$,

$$\begin{aligned}
F_i(N_0,C,G) &= \frac{1}{|G^k|!}\sum_{\pi\in X^k_1}\left[v_{(C^\varphi)^*}\left(Pre(i,\pi)\cup\{i\}\right) - v_{(C^\varphi)^*}\left(Pre(i,\pi)\right)\right]\\
&\quad + \frac{1}{|G^k|!}\sum_{\pi\in X^k_2}\left[v_{(C^\varphi)^*}\left(Pre(i,\pi)\cup\{i\}\right) - v_{(C^\varphi)^*}\left(Pre(i,\pi)\right)\right].
\end{aligned}$$

By Claim 6.2,

$$\frac{1}{|G^k|!}\sum_{\pi\in X_2^k}\left[v_{(C^\varphi)^*}\left(Pre\left(i,\pi\right)\cup\{i\}\right)-v_{(C^\varphi)^*}\left(Pre\left(i,\pi\right)\right)\right] = \frac{1}{|G^k|!}\left|X_2^k\right|\varphi_k\left(M_0,C^G\right)$$
$$= \frac{1}{|G^k|}\varphi_k\left(M_0,C^G\right).$$

We know that

$$Ow_i\left(N,v_{C^*},G\right)=\frac{1}{|\Pi^G|}\sum_{\pi\in\Pi^G}\left[v_{C^*}\left(Pre\left(i,\pi\right)\cup\{i\}\right)-v_{C^*}\left(Pre\left(i,\pi\right)\right)\right].$$

Let X_1, X_2 the partition of Π^G where

$$X_1 = \left\{\pi\in\Pi^G : Pre\left(i,\pi\right)\cap G^k\neq\varnothing\right\} \text{ and}$$
$$X_2 = \left\{\pi\in\Pi^G : Pre\left(i,\pi\right)\cap G^k=\varnothing\right\}.$$

Since $\left|\Pi^G\right|=m!\left|G^k\right|!$,

$$Ow_i\left(N,v_{C^*},G\right) = \frac{1}{m!\left|G^k\right|!}\sum_{\pi\in X_1}\left[v_{C^*}\left(Pre\left(i,\pi\right)\cup\{i\}\right)-v_{C^*}\left(Pre\left(i,\pi\right)\right)\right]$$
$$+\frac{1}{m!\left|G^k\right|!}\sum_{\pi\in X_2}\left[v_{C^*}\left(Pre\left(i,\pi\right)\cup\{i\}\right)-v_{C^*}\left(Pre\left(i,\pi\right)\right)\right].$$

By Claim 5,

$$\frac{1}{m!\left|G^k\right|!}\sum_{\pi\in X_1}\left[v_{C^*}\left(Pre\left(i,\pi\right)\cup\{i\}\right)-v_{C^*}\left(Pre\left(i,\pi\right)\right)\right]$$

$$=\frac{1}{m!\left|G^k\right|!}\sum_{\pi\in X_1}\left[v_{(C^\varphi)^*}\left(Pre\left(i,\pi_{G^k}\right)\cup\{i\}\right)-v_{(C^\varphi)^*}\left(Pre\left(i,\pi_{G^k}\right)\right)\right].$$

For each $\pi^k\in\Pi_{G^k}$, $\left|\left\{\pi\in X_1:\pi_{G^k}=\pi^k\right\}\right|=m!$. Thus, the last expression coincides with

$$=\frac{1}{|G^k|!}\sum_{\pi^k\in X_1^k}\left[v_{(C^\varphi)^*}\left(Pre\left(i,\pi^k\right)\cup\{i\}\right)-v_{(C^\varphi)^*}\left(Pre\left(i,\pi^k\right)\right)\right].$$

Let Π_G denote the set of all orders of the m coalitions $\{G_1, ..., G_m\}$. Given $\pi \in \Pi_N$, let π' denote the order induced by π among the coalitions (as in Claim 6). For each $\pi_G \in \Pi_G$,

$$|\{\pi \in X_2 : \pi' = \pi_G\}| = \left(\left|G^k\right| - 1\right)!$$

By Claim 6.1,

$$\begin{aligned}
& \frac{1}{m! \, |G^k|!} \sum_{\pi \in X_2} \left[v_{C^*} \left(Pre\left(i, \pi\right) \cup \{i\}\right) - v_{C^*} \left(Pre\left(i, \pi\right)\right)\right] \\
= \; & \frac{1}{m! \, |G^k|!} \sum_{\pi \in X_2} \left[v_{(C^G)^*} \left(Pre\left(k, \pi'\right) \cup \{k\}\right) - v_{(C^G)^*} \left(Pre\left(k, \pi'\right)\right)\right] \\
= \; & \frac{1}{m! \, |G^k|} \sum_{\pi' \in \Pi_G} \left[v_{(C^G)^*} \left(Pre\left(k, \pi'\right) \cup \{k\}\right) - v_{(C^G)^*} \left(Pre\left(k, \pi'\right)\right)\right] \\
= \; & \frac{1}{|G^k|} \varphi_k \left(M_0, C^G\right).
\end{aligned}$$

Then, $F_i\left(N_0, C, G\right) = Ow_i\left(N, v_{C^*}, G\right)$. ■

And the proof of Theorem 3 is completed ■

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