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Gustavo Bergantiños and María Gómez-Rúa.

**Report 08/06**

**Discussion Papers in Statistics and Operation Research**

Departamento de Estatística e Investigación Operativa

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# Minimum cost spanning tree problems with coalition structure*

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**Abstract**

We study minimum cost spanning tree problems with coalition structure. We assume that agents are located in different villages. We introduce a rule for dividing the cost of connecting all agents to the source among the agents taking into account the coalition structure. We characterize this rule.

## 1 Introduction

In this paper we study minimum cost spanning tree problems (*mcstp*). A group of agents (denoted by $N$), located at different geographical places, want a particular service which can only be provided by a common supplier, called the source (denoted by 0). Agents will be served through connections which involve some cost. Moreover, they do not care whether they are connected directly or indirectly to the source. This situation is described by a symmetric matrix $C$, where $c_{ij}$ denotes the connection costs between $i$ and $j$ $(i, j \in N \cup \{0\})$.

There are many economic situations that can be modeled in this way. For instance, several towns may draw power from a common power plant, and hence have to share the cost of the distribution network (Dutta and Kar, 2004). Bergantiños and Lorenzo (2004, 2005) study a real situation where villagers had to pay the cost of constructing pipes from their respective houses

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to a water supplier. Other examples include communication networks, such as telephone, internet, or cable television.

We assume that agents construct a minimum cost spanning tree ($mt$). The question is how to divide the cost associated with the $mt$ between the agents. Different rules give different answers to this question. One of the most important topics is the axiomatic characterization of rules. The idea is to propose desirable properties and to find out which of them characterize each rule. Properties often help agents to compare different rules and to decide which rule is preferred in a particular situation.

In some cases, as in Dutta and Kar (2004) or Bergantiños and Lorenzo (2004, 2005), agents are located in different villages. This means, in terms of the cost matrix, that the connection cost between two agents of the same village is not larger than the connection cost between an agent of this village and an agent from other village.

The classical model of $mcstp$, as described above, can also model these situations. Nevertheless, it ignores the fact that some group of agents are located in the same city or village. It could be interesting to include this fact in the model. We do it by considering an extra element in the model. Namely a partition $G = \{G^1, ..., G^m\}$ of the set of agents $N$. For each $k = 1, ..., m$, $G^k$ represents the coalition of agents located in the same village, city, ...

In this paper we follow the axiomatic approach and we introduce a rule as the unique rule satisfying a set of desirable properties. Our idea is to generalize the axiomatic characterization of the rule $\varphi$ given by Bergantiños and Vidal-Puga (2007c), which involves three properties. Restricted Additivity ($RA$) which says that the rule must be additive on the cost matrix; Population Monotonicity ($PM$), which says that if a new agent comes, no agent of the initial society can be worse off; and Symmetry ($SYM$), which says that symmetric agents (with respect to the cost matrix) must pay the same.

We adapt these properties to $mcstp$ with coalition structure. The property of $RA$ could be formulated in a similar way. Nevertheless, $PM$ and $SYM$ should be adapted. The main idea for adapting each of these properties is claiming both twice. First among the coalitions and then among agents inside the same coalition.

In order to adapt these properties a question comes to our mind. Should the cost paid by the agents of a village depend on the internal characteristics of the other village? For instance, should this cost depend on the number of agents of the other villages? We consider that both answers, "yes" or "no", are reasonable. In this paper we have chosen "no". Then, we have adapted the properties of $PM$ and $SYM$ taking it into account.

We consider two properties of $SYM$. Symmetry among agents in the same coalition ($SYMA$) says that if two agents are symmetric and belong to the same coalition, they must pay the same. Symmetry among coalitions ($SYMC$) says that if two coalitions are symmetric the total amount paid by the members of each coalition minus the cost of connecting agents inside the coalition among themselves must be the same. Two coalitions are symmetric if their connection costs to the other coalitions are the same.

We also consider two properties of $PM$. Population monotonicity over agents ($PMA$) says that if agent $i$ enters coalition $G^k$, no agent of coalition $G^k$ can be worse off. Moreover, if the connection costs between coalition $G^k$ and the other coalitions do not change, agents of the other coalitions must pay the same. Population monotonicity over coalitions ($PMC$) says that if a new coalition joins the society, no agent of the initial society can be worse off.

The main result of the paper says that there is a unique rule, we call it $F$, satisfying $RA$, $SYMA$, $SYMC$, $PMA$, and $PMC$.

We now describe the rule $F$. $F$ can be considered as a two-steps rule. In the first step we compute the amount that each coalition should pay in order to be connected to the source. We do it applying the rule $\varphi$ defined in Bergantiños and Vidal-Puga (2007a). In the second step we decide the amount that each agent of each coalition has to pay. For each coalition $G^k$, we consider the *mcstp* inside each coalition $\left(G_0^k, C^\varphi\right)$. In $\left(G_0^k, C^\varphi\right)$ the connection cost between two agents in $G^k$ is the same as in $C$. Nevertheless, the connection cost between any agent of $G^k$ and the source is the amount computed for the coalition $G^k$ in the first step.

The paper is organized as follows. In Section 2 we introduce *mcstp*. In Section 3 we introduce *mcstp* with coalition structure. In Section 4 we define the rule $F$ and we present the axiomatic characterization.

# 2 Minimum cost spanning tree problems

Let $\mathcal{N} = \{1, 2, ...\}$ be the set of all possible agents. Given a finite set $N \subset \mathcal{N}$, let $\Pi_N$ be the set of all permutations over $N$. Given $\pi \in \Pi_N$, let $Pre\,(i, \pi)$ denote the set of the elements of $N$ which come before $i$ in the order given by $\pi$, *i.e.* $Pre\,(i, \pi) = \{j \in N : \pi\,(j) < \pi\,(i)\}$. Given $S \subset N$, let $\pi_S$ denote the order induced by $\pi$ among the agents in $S$.

We are interested in networks whose nodes are elements of a set $N_0 = N \cup \{0\}$, where $N \subset \mathcal{N}$ is finite and 0 is a special node called the *source*. Usually we take $N = \{1, ..., n\}$.

A *cost matrix* $C = (c_{ij})_{i,j \in N_0}$ on $N$ represents the cost of direct link between any pair of nodes. We assume that $c_{ij} = c_{ji} \geq 0$ for each $i, j \in N_0$

and $c_{ii} = 0$ for each $i \in N_0$. Since $c_{ij} = c_{ji}$ we work with undirected arcs, *i.e.* $(i, j) = (j, i)$.

We denote the set of all cost matrices over $N$ as $\mathcal{C}^N$. Given $C$, $C' \in \mathcal{C}^N$ we say $C \leq C'$ if $c_{ij} \leq c'_{ij}$ for all $i, j \in N_0$.

A *minimum cost spanning tree problem*, briefly an *mcstp*, is a pair $(N_0, C)$ where $N \subset \mathcal{N}$ is a finite set of agents, 0 is the source, and $C \in \mathcal{C}^N$ is the cost matrix.

Given an *mcstp* $(N_0, C)$, we define the *mcstp* induced by $C$ in $S \subset N$ as $(S_0, C)$.

A *network* $g$ over $N_0$ is a subset of $\{(i, j) : i, j \in N_0\}$. The elements of $g$ are called *arcs*. Given a network $g$ over $N_0$ and $S \subset N_0$ we denote by $g_S$ the network induced by $g$ among the elements of $S$. Namely, $g_S = \{(i, j) \in g : \{i, j\} \subset S\}$.

Given a network $g$ and a pair of nodes $i$ and $j$, a *path* from $i$ to $j$ in $g$ is a sequence of different arcs $\{(i_{h-1}, i_h)\}_{h=1}^{l}$ satisfying $(i_{h-1}, i_h) \in g$ for all $h \in \{1, 2, ..., l\}$, $i = i_0$, and $j = i_l$.

A *tree* is a network such that for all $i \in N$ there is a unique path from $i$ to the source. If $t$ is a tree, we usually write $t = \{(i^0, i)\}_{i \in N}$ where $i^0$ represents the first agent in the unique path in $t$ from $i$ to 0.

Let $\mathcal{G}^N$ denote the set of all networks over $N_0$. Let $\mathcal{G}_0^N$ denote the set of all networks where every agent $i \in N$ is connected to the source, *i.e.* there exists a path from $i$ to 0 in the network.

Given an *mcstp* $(N_0, C)$ and $g \in \mathcal{G}^N$, we define the *cost* associated with $g$ as

$$c(N_0, C, g) = \sum_{(i,j) \in g} c_{ij}.$$

When there is no ambiguity, we write $c(g)$ or $c(C, g)$ instead of $c(N_0, C, g)$.

A *minimum cost spanning tree* for $(N_0, C)$, briefly an *mt*, is a tree $t$ over $N_0$ such that $c(t) = \min_{g \in \mathcal{G}_0^N} c(g)$. It is well-known that an *mt* exists, even though it is not necessarily unique. Given an *mcstp* $(N_0, C)$, we denote the cost associated with any *mt* as $m(N_0, C)$.

Given an *mcstp*, Prim (1957) provides an algorithm for solving the problem of connecting all agents to the source such that the total cost of creating the network is minimal. The idea of this algorithm is simple: starting from the source we construct a network by sequentially adding arcs with the lowest cost and without introducing cycles.

Formally, Prim's algorithm is defined as follows. We start with $S^0 = \{0\}$ and $g^0 = \emptyset$.

*Stage 1*: Take an arc $(0, i_1)$ such that $c_{0i_1} = \min\limits_{j \in N} \{c_{0j}\}$. If there are several arcs satisfying this condition, select just one. Now, $S^1 = \{0, i_1\}$ and $g^1 = \{(0, i_1)\}$.

*Stage* $p+1$: Assume that we have defined $S^p \subset N_0$ and $g^p \in \mathcal{G}^N$. We now define $S^{p+1}$ and $g^{p+1}$. Take an arc $\left(i^0_{p+1}, i_{p+1}\right)$ with $i^0_{p+1} \in S^p$ and $i_{p+1} \in N_0 \backslash S^p$ such that $c_{i^0_{p+1} i_{p+1}} = \min\limits_{k \in S^p, l \in N_0 \backslash S^p} \{c_{kl}\}$. If there are several arcs satisfying this condition, select just one. Now, $S^{p+1} = S^p \cup \{i_{p+1}\}$ and $g^{p+1} = g^p \cup \left\{\left(i^0_{p+1}, i_{p+1}\right)\right\}$.

This process is completed in $n$ stages. We say that $g^n$ is a tree obtained following Prim's algorithm. Notice that this algorithm leads to a tree, but this is not always unique.

Given an *mcstp* $(N_0, C)$ and an *mt* $t$, Bird (1976) defined the *minimal network* $(N_0, C^t)$ associated with $t$ as follows: $c^t_{ij} = \max\limits_{(k,l) \in g_{ij}} \{c_{kl}\}$, where $g_{ij}$ denotes the unique path in $t$ from $i$ to $j$. Even though $g_{ij}$ depends on the choice of $t$, $c^t_{ij}$ is independent of the chosen $t$. Proof of this can be found, for instance, in Aarts and Driessen (1993).

The *irreducible form* of an *mcstp* $(N_0, C)$ is defined as the minimal network $(N_0, C^*)$ associated with a particular *mt* $t$. If $(N_0, C^*)$ is an *irreducible form*, we say that $C^*$ is an *irreducible matrix*.

One of the most important issues addressed in the literature about *mcstp* is how to divide the cost of connecting agents to the source between them. We now briefly describe some of the rules studied in the literature.

A (*cost allocation*) *rule* is a function $f$ such that $f(N_0, C) \in \mathbb{R}^N$ and $\sum\limits_{i \in N} f_i(N_0, C) = m(N_0, C)$ for each *mcstp* $(N_0, C)$. As usual, $\psi_i(N_0, C)$ represents the cost allocated to agent $i$.

Notice that we implicitly assume that the agents build an *mt*. As far as we know, all the rules proposed in the literature make this assumption.

A *coalitional game with transferable utility*, briefly a *TU game*, is a pair $(N, v)$ where $v : 2^N \to \mathbb{R}$ satisfies $v(\varnothing) = 0$. $Sh(N, v)$ denotes the Shapley value (Shapley (1953)) of $(N, v)$.

For each *mcstp* $(N_0, C)$, Bird (1976) introduces the *TU* game $(N, v_C)$. For each coalition $S \subset N$,

$$v_C(S) = m(S_0, C).$$

There are several rules studied in the literature. We mention, for instance, the rules studied in Bird (1976), Kar (2002), and Dutta and Kar (2004). In this paper the rule introduced by Feltkamp *et al* (1994) and called Equal Remaining Obligations rule ($ERO$) will be very important. $ERO$ is called the *P-value* in Branzei *et al* (2004).

On the other hand, in Bergantiños and Vidal-Puga (2007a) it is defined the rule $\varphi$ as

$$\varphi(N_0, C) = Sh(N, v_{C^*})$$

where $C^*$ is the irreducible matrix associated with $C$. Bergantiños and Vidal-Puga (2007e) prove that, surprisingly, $\varphi$ coincides with $ERO$. This rule is also studied in Bergantiños and Vidal-Puga (2007b, 2007c, 2007d).

We now define several properties formally.

We say that $f$ satisfies *Restricted Additivity* ($RA$) if for all *mcstp* $(N_0, C)$ and $(N_0, C')$ satisfying that there exists an *mt* $t = \{(i^0, i)\}_{i \in N}$ in $(N_0, C)$, $(N_0, C')$, and $(N_0, C + C')$ and an order $\pi = (i_1, \ldots, i_{|N|}) \in \Pi_N$ such that $c_{i_1^0 i_1} \leq c_{i_2^0 i_2} \leq \ldots \leq c_{i_{|N|}^0 i_{|N|}}$ and $c'_{i_1^0 i_1} \leq c'_{i_2^0 i_2} \leq \ldots \leq c'_{i_{|N|}^0 i_{|N|}}$, we have that

$$f(N_0, C + C') = f(N_0, C) + f(N_0, C').$$

$RA$ is an additivity property restricted to some subclass of problems. No rule satisfies additivity over all *mcstp*. The reason is that in the definition of a rule we are claiming that $\sum_{i \in N} f_i(N_0, C) = m(N_0, C)$, which is incompatible with additivity over all *mcstp*. See Bergantiños and Vidal-Puga (2007c) for a detailed discussion of $RA$.

We say that $f$ satisfies *Population Monotonicity* ($PM$) if for all *mcstp* $(N_0, C)$, all $S \subset N$, and all $i \in S$,

$$f_i(N_0, C) \leq f_i(S_0, C).$$

$PM$ says that, if new agents join a society, no agent of the initial society can be worse off. This is a well-known property, which has been used in many different situations.

We say that $i, j \in N$ are *symmetric* if for all $k \in N_0 \setminus \{i, j\}$, $c_{ik} = c_{jk}$.

We say that $f$ satisfies *Symmetry* ($SYM$) if for all *mcstp* $(N_0, C)$ and all pair of symmetric agents $i, j \in N$,

$$f_i(N_0, C) = f_j(N_0, C).$$

We say that $f$ satisfies *Strong Cost Monotonicity* ($SCM$) if for all *mcstp* $(N_0, C)$ and $(N_0, C')$ such that $C \leq C'$ and all $i \in N$,

$$f_i(N_0, C) \leq f_i(N_0, C').$$

$SCM$ implies that if a number of connection costs increase and the rest of connection cost (if any) remain the same, no agent can be better off. This property is called *solidarity* in Bergantiños and Vidal-Puga (2007a).

In Lemma 1 below we present some results used in the paper. The proof can be found in Bergantiños and Vidal-Puga (2007a, 2007b, 2007c, 2007d).

**Lemma 1** *(a) $(N_0, C)$ is irreducible if and only if there exists an mt $t$ in $(N_0, C)$ satisfying the two following conditions:*

*$(A1)$ $t = \{(i_{p-1}, i_p)\}_{p=1}^{|N|}$ where $i_0 = 0$.*

*$(A2)$ Given $i_p, i_q \in N_0$, $p < q$, then $c_{i_p i_q} = \max_{p<r\leq q} \left\{c_{i_{r-1} i_r}\right\}$.*

*(b) If $C$ is an irreducible matrix, then for all $S \subset N_0$, $i \notin S$ we have that*

$$v_C(S \cup \{i\}) - v_C(S) = \min_{j \in S_0} \{c_{ij}\}.$$

*(c) If $C$ is an irreducible matrix, then $v_C$ is a concave game. Namely, if $S \subset T \subset N$ and $i \notin T$, then*

$$v_C(S \cup \{i\}) - v_C(S) \geq v_C(T \cup \{i\}) - v_C(T).$$

*(d) If $C$ and $C'$ are under the conditions of $RA$, then for all $S \subset N$,*

$$v_{(C+C')^*}(S) = v_{C^*}(S) + v_{C'^*}(S).$$

*(e) $\varphi$ is the unique rule on mcstp satisfying $RA$, $PM$, and $SYM$.*

*(f) $\varphi$ satisfies $SCM$.*

# 3 Minimum cost spanning tree problems with coalition structure

There are many economic situations that can be modeled as a *mcstp*. Let us mention some examples. Several towns may draw power from a common power plant, and hence have to share the cost of the distribution network (Dutta and Kar, 2004). Bergantiños and Lorenzo (2004, 2005) study a real situation where a valley authority has to construct pipes from a dam to several houses. These houses are located in different villages of the valley.

The classical model of *mcstp*, as described in the previous section can also model this situation. Nevertheless, it ignores the fact that some groups of agents are located in the same city or village. It could be interesting to include this fact in the model. That is the main issue of this section.

We do it by considering an extra element in the model. Namely a partition $G = \{G^1, ..., G^m\}$ of the set of agents $N$. The interpretation of $G$ is clear. For each $k = 1, ..., m$, $G^k$ represents a coalition of agents, which are located in the same city, village, ...

In many situations, for instance the examples mentioned at the beginning of this section, the cost between any pair of agents is closely related to the distance between both agents. Under these circumstances, it seems reasonable that the connection cost between two agents of city $G^k$ is not larger than the connection cost between an agent of city $G^k$ and an agent from another city (or the source).

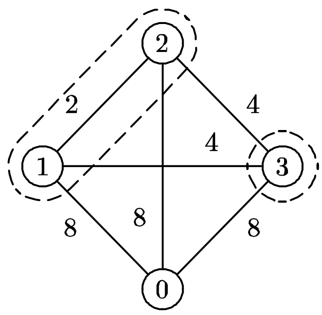
We now introduce the model formally. An *mcstp* with coalition structure is a triple $(N_0, C, G)$ where $(N_0, C)$ is an *mcstp*, $G = \{G^1, ..., G^m\}$ is a partition of $N$ and for each $k = 1, ...m$

$$\max_{i,j \in G^k} \{c_{ij}\} \leq \min_{i \in G^k, j \notin G^k} \{c_{ij}\}.$$

A *rule* in *mcstp* with coalition structure is a function $f$ such that $f(N_0, C, G) \in \mathbb{R}^N$ and $\sum_{i \in N} f_i(N_0, C, G) = m(N_0, C)$ for each *mcstp* $(N_0, C)$.

As in classical *mcstp*, the main objective is to divide the cost associated with an *mt* among the agents in a fair way.

**Example 2** *Consider the mcstp $(N_0, C, G)$ with coalition structure where $N = \{1, 2, 3\}$, $G = \{G^1, G^2\}$, $G^1 = \{1, 2\}$, $G^2 = \{3\}$, and matrix $C$ which is represented in the following figure:*




*In this case $m(N_0, C) = 14$. Moreover, 12 units are associated with the cost of connecting cities 1 and 2 with the source and 2 units are associated with the cost of connecting agents 1 and 2 inside city 1.*

*Since we are looking for fair shares it seems reasonable to divide these 2 units equally between agents 1 and 2.*

*The 12 units comes from the construction of the network in which some of the three agents is connected with the source and some agent of $G^1$ is connected with agent* 3. *In order to divide the 12 units among the agents two approaches seems reasonable.*

1. *The cost paid by each city does not depend on the characteristics of the other city. Assuming it both cities are symmetric. Thus, each city should pay 6. Since agents inside city 1 are also symmetric, both pay the same. Then, agent 1 pays 1+3=4, agent 2 pays 1+3=4, agent 3 pays 6.*

2. *The cost paid by each city should take into account the number of agents who get benefits from their connection. Thus, city 1 should pay twice than city 2, i.e. city 1 pays 8 and city 2 pays 4. Since agents inside city 1 are also symmetric, both pay the same. Then, agent 1 pays 1+4=5, agent 2 pays 1+4=5, agent 3 pays 4.*

In this paper we have decided to follow the first approach. Thus, some properties introduced later will be defined accordingly.

# 4 The rule and the axiomatic characterization

In this Section we follow the axiomatic approach and introduce a rule as the unique rule satisfying a set of desirable properties. Our idea is to generalize the axiomatic characterization of the rule $\varphi$ given by Bergantiños and Vidal-Puga (2007d), which involves three properties: $RA$, $PM$, and $SYM$.

Now we adapt these properties to *mcstp* with coalition structure. The property of $RA$ could be formulated in a similar way. Nevertheless, $PM$ and $SYM$ should be adapted. The main idea for adapting each of these properties is claiming both twice. Once among the coalitions and other among agents inside the same coalition.

We say that $f$ satisfies *Restricted Additivity* ($RA$) if for all *mcstp* with coalition structure $(N_0, C, G)$ and $(N_0, C', G)$ satisfying that there exists an *mt* $t = \{(i^0, i)\}_{i \in N}$ in $(N_0, C, G)$, $(N_0, C', G)$, and $(N_0, C + C', G)$ and an order $\pi = (i_1, \ldots, i_{|N|}) \in \Pi_N$ such that $c_{i_1^0 i_1} \leq c_{i_2^0 i_2} \leq \ldots \leq c_{i_{|N|}^0 i_{|N|}}$ and $c'_{i_1^0 i_1} \leq c'_{i_2^0 i_2} \leq \ldots \leq c'_{i_{|N|}^0 i_{|N|}}$, we have that

$$f(N_0, C + C', G) = f(N_0, C, G) + f(N_0, C', G).$$

We say that $f$ satisfies *Symmetry among Agents in the same Coalition* ($SYMA$) if for all *mcstp* with coalition structure $(N_0, C, G)$ and all pair of symmetric agents $i, j \in G^k \in G$,

$$f_i(N_0, C, G) = f_j(N_0, C, G).$$

We now define symmetry among coalitions. We first define symmetric coalitions. Intuitively two coalitions of agents are symmetric if their connection costs to the other coalitions are the same. Because of the model each pair of coalitions $G^k$ and $G^{k'}$ can connect in several ways. For each pair of agents $i \in G^k$, $j \in G^{k'}$ they can construct the arc $(i, j)$. Since we are assuming that agents will construct an *mt*, it is reasonable to assume that they will construct an arc $(i, j)$ with minimum cost.

We say that *two coalitions $G^k$ and $G^{k'}$ are symmetric* if for all $G^l \in G_0 \setminus \{G^k, G^{k'}\}$,

$$\min_{i \in G^k, j \in G^l} \{c_{ij}\} = \min_{i \in G^{k'}, j \in G^l} \{c_{ij}\}.$$

The next step is to say that symmetric coalitions should pay the same. The amount paid by coalition $G^k$ is $\sum_{i \in G^k} f_i(N_0, C, G)$. Thus, we can decompose this amount in two parts: the cost of connecting agents inside the coalition among themselves, $m(G^k, C)$, and the cost of connecting the coalition with the source (possibly through other coalitions), $\sum_{i \in G^k} f_i(N_0, C, G) - m(G^k, C)$. We are assuming that the amount paid by a coalition should not depend on the internal characteristics of the other coalitions. Then, it seems reasonable to say that $m(G^k, C)$ should be paid by agents of $G^k$. Thus, we formulate the second symmetry property as follows.

We say that $f$ satisfies *Symmetry among Coalitions* ($SYMC$) if for all *mcstp* with coalition structure $(N_0, C, G)$ and all pair symmetric coalitions $G^k, G^{k'} \in G$,

$$\sum_{i \in G^k} f_i(N_0, C, G) - m(G^k, C) = \sum_{i \in G^{k'}} f_i(N_0, C, G) - m\left(G^{k'}, C\right).$$

We now define the two population monotonicity properties, over coalitions and over agents.

The idea of population monotonicity over coalitions is quite simple. If a new coalition joins the society, no agent in the initial society can be worse off. Formally, we say that $f$ satisfies *Population Monotonicity over Coalitions* $(PMC)$ if for all *mcstp* with coalition structure $(N_0, C, G)$, all $G^k \in G$, and all $i \in N \backslash G^k$,

$$f_i(N_0, C, G) \leq f_i\left(\left(N \backslash G^k\right)_0, C, G \backslash G^k\right).$$

The population monotonicity over agents will apply when an agent enters a coalition. We claim that no agent in the *initial coalition* can be worse off.

Furthermore, assume that after the entrance of agent $i$ in coalition $G^k$ the minimum connection cost between coalition $G^k$ and the rest of the coalitions does not change, *i.e.* for each $G^l$, $l \neq k$, $\min\limits_{j \in G^k, j' \in G^l} \{c_{jj'}\} = \min\limits_{j \in G^k \cup \{i\}, j' \in G^l} \{c_{jj'}\}$. Since we are assuming that the amount paid by a coalition should not depend on the internal characteristics of the other coalitions, and the entrance of agent $i$ does not change the connection cost among coalitions, we claim that the agents in the others coalitions must pay the same.

Formally, we say that $f$ satisfies *Population Monotonicity over Agents* $(PMA)$ if for all *mcstp* with coalition structure $(N_0, C, G)$, all $G^k \in G$, and all $i \in G^k$ such that $G^k \backslash \{i\} \neq \varnothing$,

$$f_j(N_0, C, G) \leq f_j\left((N \backslash \{i\})_0, C, \left(G \backslash G^k\right) \cup \left(G^k \backslash \{i\}\right)\right)$$

for all $j \in G^k \backslash \{i\}$.

Moreover, if for each $G^l$ with $l \neq k$, $\min\limits_{j \in G^k, j' \in G^l} \{c_{jj'}\} = \min\limits_{j \in G^k \backslash \{i\}, j' \in G^l} \{c_{jj'}\}$, then

$$f_j(N_0, C, G) = f_j\left((N \backslash \{i\})_0, C, \left(G \backslash G^k\right) \cup \left(G^k \backslash \{i\}\right)\right)$$

for all $j \in N \backslash G^k$.

We now define the rule $F$ in *mcstp* with coalition structure. We first give the intuitive idea. This rule can be considered as a two-steps rule. In the first step we compute the amount that each coalition should pay in order to be connected to the source. We do it applying the rule $\varphi$ defined in Bergantiños and Vidal-Puga (2007a).

In the second step we decide the amount that each agent of each coalition has to pay. For each coalition $G^k$, we consider the *mcstp* inside each coalition $\left(G_0^k, C^\varphi\right)$. In this *mcstp*, the connection cost between two agents of $G^k$ is the same as in $C$ but the connection cost between any agent of $G^k$ and the source is the amount computed by the coalition $G^k$ in the first step.

We now present the definition formally. Given the *mcstp* with coalition structure $(N_0, C, G)$, with $G = \{G^1, ..., G^m\}$ and $M = \{1, ..., m\}$ we define the *mcstp* among coalitions $\left(M_0, C^G\right)$ as follows:

- $M_0 = \{0, 1, ..., m\}$.
- $C^G$ is the cost matrix and for each $k, k' \in M_0$ the connection cost between $k$ and $k'$ is denoted by

$$c_{kk'}^G = \min_{i \in G^k, j \in G^{k'}} \{c_{ij}\}.$$

Let $(N_0, C, G)$ be an *mcstp* with coalition structure and $i \in G^k$. We define,

$$F_i(N_0, C, G) = \varphi_i\left(G_0^k, C^\varphi\right)$$

where

$$c_{jj'}^\varphi = \begin{cases} c_{jj'} & \text{if } 0 \notin \{j, j'\} \\ \varphi_k\left(M_0, C^G\right) & \text{if } 0 \in \{j, j'\}. \end{cases}$$

Before introducing the results of the paper, we present Lemma 3, which will be used often in the proofs of the main results.

**Lemma 3** *Given $(N_0, C, G)$ we can find an mt $t$ in $(N_0, C)$ satisfying:*
*$(i)$ For each $k = 1, ..., m$, $t_{G^k}$ induces an mt in $\left(G^k, C\right)$.*
*$(ii)$ $\left\{(k, k') : \exists i \in G^k, j \in G^{k'} \text{ with } (i, j) \in t\right\}$ is an mt in $\left(M_0, C^G\right)$.*
*$(iii)$ For each $k = 1, ...., m$ and each $i \in G^k$, $t_{G_k} \cup \{(0, i)\}$ is an mt in $(G_0^k, C^\varphi)$.*

***Proof.*** *We prove that, when we apply Prim's algorithm, if at Stage $p$, $S^p = \left(\bigcup_{j=1}^{l} G^{k_j}\right) \cup G'$ where $G' \subset G^{k_{l+1}}$, $G' \neq \varnothing$, and $G' \neq G^{k_{l+1}}$, then at Stage $p+1$ we can select an arc $\left(i_{p+1}^0, i_{p+1}\right)$ where $i_{p+1}^0 \in G'$ and $i_{p+1} \in G^{k_{l+1}} \backslash G'$.*
*Let $\left(i_{p+1}^0, i_{p+1}\right)$ be such that $i_{p+1}^0 \in G'$, $i_{p+1} \in G^{k_{l+1}} \backslash G'$ and*

$$c_{i_{p+1}^0 i_{p+1}} = \min_{i \in G', j \in G^{k_{l+1}} \backslash G'} \{c_{ij}\}.$$

*By definition of Prim's algorithm it is enough to prove that $c_{i_{p+1}^0 i_{p+1}} \leq c_{ij}$ in the following cases:*

1. *$i \in G'$, $j \in N \backslash \left(\bigcup_{j=1}^{l+1} G^{k_j}\right)$. Thus, $c_{i_{p+1}^0 i_{p+1}} \leq c_{ij}$ because $\left\{i_{p+1}^0, i_{p+1}\right\} \subset G^{k_{l+1}}$, $i \in G^{k_{l+1}}$, $j \in G^{k'}$, $k' \in \{1, ..., m\} \setminus \{k_1, ..., k_{l+1}\}$, and $k_{l+1} \neq k'$.*

2. $i \in \bigcup_{j=1}^{l} G^{k_j}$, $j \in G^{k_{l+1}} \backslash G'$. *Thus,* $c_{i^0_{p+1} i_{p+1}} \leq c_{ij}$ *because* $\left\{ i^0_{p+1}, i_{p+1} \right\} \subset G^{k_{l+1}}$, $i \in G^{k'}$, $k' \in \{k_1, ..., k_l\}$, $j \in G^{k_{l+1}}$, *and* $k_{l+1} \neq k'$.

3. $i \in \bigcup_{j=1}^{l} G^{k_j}$, $j \notin G^{k_{l+1}} \backslash G'$. *Thus,* $c_{i^0_{p+1} i_{p+1}} \leq c_{ij}$ *because* $\left\{ i^0_{p+1}, i_{p+1} \right\} \subset G^{k_{l+1}}$, $i \in G^{k'}, k' \in \{k_1, ..., k_l\}$, $j \in G^{k''}$, $k'' \in \{1, ..., m\} \setminus \{k_1, ..., k_{l+1}\}$ *and* $k'' \neq k'$.

*We now prove parts* $(i)$ *and* $(ii)$.

*We apply Prim's algorithm to* $(N_0, C)$. *Let* $(0, i_1)$ *be the first arc selected according Prim's algorithm. We assume wlog that* $i_1 \in G^1$. *If* $G^1 \setminus \{i_1\} \neq \varnothing$, *by the previous statement, in Stage 2 of Prim's algorithm we can select an arc* $(i^0_2, i_2)$ *satisfying that* $i^0_2 \in G^1 \cap S^1 = \{i_1\}$ *and* $i_2 \in G^1 \setminus \{i_1\}$.

*If we repeat this argument we can prove that for each* $p = 2, ..., |G^1|$ *the arc* $\left(i^0_p, i_p\right)$ *selected at Stage* $p$ *satisfies* $i^0_p \in G^1$ *and* $i_p \in G^1$.

*In Stage* $p = |G^1| + 1$ *we select an arc* $\left(i^0_p, i_p\right)$ *where* $i^0_p \in G^1 \cup \{0\}$ *and* $i_p \notin G^1 \cup \{0\}$. *We assume wlog that* $i_p \in G^2$. *By definition of Prim's algorithm,* $c_{i^0_p i_p} = c^G_{12}$ *when* $i^0_p \in G^1$ *whereas* $c_{i^0_p i_p} = c^G_{02}$ *when* $i^0_p = 0$. *Repeating the same argument as above we can prove that for each* $p = |G^1| + 2, ..., |G^1 \cup G^2|$ *the arc* $\left(i^0_p, i_p\right)$ *selected in Stage* $p$ *satisfies* $i^0_p \in G^2$ *and* $i_p \in G^2$.

*In general, for each* $q = 1, ..., m$ *and for each* $p = \left|\cup_{l=1}^{q-1} G^l\right| + 2, ..., \left|\cup_{l=1}^{q} G^l\right|$ *the arc* $\left(i^0_p, i_p\right)$ *selected in Stage* $p$ *satisfies* $i^0_p \in G^q$ *and* $i_p \in G^q$. *Moreover, for each* $q = 1, ..., m$ *and for each* $p = \left|\cup_{l=1}^{q-1} G^l\right| + 1$ *the arc* $\left(i^0_p, i_p\right)$ *selected in Stage* $p$ *satisfies* $i^0_p \in \cup_{l=1}^{q-1} G^l \cup \{0\}$ *and* $i_p \in \cup_{l=q-1}^{m} G^l$.

*Now it is easy to conclude that parts* $(i)$ *and* $(ii)$ *hold.*

*We now prove part* $(iii)$. *Let* $k \in \{1, ..., m\}$. *Because of parts* $(i)$ *and* $(ii)$ *it is enough to prove that for each* $i \in G^k$ $c^{\varphi}_{0i} \geq \max_{j,j' \in G^k} \left\{ c^{\varphi}_{jj'} \right\}$. *Since* $c^{\varphi}_{0i} = \varphi_k(M_0, C^G)$ *for all* $i \in G^k$ *and* $c^{\varphi}_{jj'} = c_{jj'}$ *for all* $j, j' \in G^k$, *we must prove that* $\varphi_k(M_0, C^G) \geq \max_{j,j' \in G^k} \{c_{jj'}\}$.

*Given an mcstp* $(N_0, C)$, *for all* $i \in N$, $\varphi_i(N_0, C) = Sh_i(N, v_{C^*})$. *By Lemma 1* $(b)$, *for all* $S \subset N$, $i \notin S$, $v_{C^*}(S \cup \{i\}) - v_{C^*}(S) = c^*_{ij}$ *for some* $j \in S_0 \setminus \{i\}$. *So,* $\varphi_i(N_0, C) \geq \min_{j \in N_0 \setminus \{i\}} \left\{ c^*_{ij} \right\}$.

*Thus,* $\varphi_k(M_0, C^G) \geq \min_{k' \in M_0 \setminus \{k\}} \left\{ \left(c^G_{kk'}\right)^* \right\}$. *Since the matrix irreducible is the minimal network associated with an mt,*

$$\min_{k' \in M_0 \setminus \{k\}} \left\{ \left(c^G_{kk'}\right)^* \right\} \geq \min_{k' \in M_0 \setminus \{k\}} \left\{ c^G_{kk'} \right\}.$$

*Because of the definition of* $(N_0, C, G)$, $\min_{k' \in M_0 \setminus \{k\}} \left\{ c^G_{kk'} \right\} \geq \max_{j,j' \in G^k} \{ c_{jj'} \}$, *Thus, Lemma 3* $(iii)$ *holds.* ■

In the next Proposition we prove that $F$ is a rule in *mcstp* with coalition structure. We also prove that $F$ generalizes the rule $\varphi$ defined in Bergantiños and Vidal-Puga (2007a).

**Proposition 4** $(a)$ *For each mcstp with coalition structure* $(N_0, C, G)$,

$$\sum_{i \in N} F_i (N_0, C, G) = m (N_0, C).$$

$(b)$ *Let* $(N_0, C, G)$ *be an mcstp with coalition structure where* $G = \{N\}$. *Then, for each* $i \in N$,

$$F_i (N_0, C, G) = \varphi_i (N_0, C).$$

$(c)$ *Let* $(N_0, C, G)$ *be an mcstp with coalition structure where* $G = \{\{i\}\}_{i \in N}$. *Then, for each* $i \in N$,

$$F_i (N_0, C, G) = \varphi_i (N_0, C).$$

***Proof.*** $(a)$ *By definition of* $F$,

$$\sum_{i \in N} F_i (N_0, C, G) = \sum_{k=1}^{m} \sum_{i \in G^k} \varphi_i \left( G^k_0, C^\varphi \right).$$

*Since* $\varphi$ *is a rule in mcstp, for each* $k = 1, ..., m$, $\sum_{i \in G^k} \varphi_i \left( G^k_0, C^\varphi \right) = m(G^k_0, C^\varphi)$. *So,*

$$\sum_{i \in N} F_i (N_0, C, G) = \sum_{k=1}^{m} m(G^k_0, C^\varphi).$$

*By Lemma 3* $(iii)$, *for any* $k = 1, ..., m$, *we can construct an mt in* $(G^k_0, C^\varphi)$, $t_{G^k} \cup \{(0, i)\}$ *with* $i \in G^k$. *Thus,* $m(G^k_0, C^\varphi) = m(G^k, C^\varphi) + \varphi_k(M_0, C^G)$. *Hence,*

$$\begin{aligned} \sum_{i \in N} F_i (N_0, C, G) &= \sum_{k=1}^{m} \left[ m(G^k, C^\varphi) + \varphi_k(M_0, C^G) \right] \\ &= \sum_{k=1}^{m} m(G^k, C^\varphi) + \sum_{k=1}^{m} \varphi_k(M_0, C^G). \end{aligned}$$

*Since $\varphi$ is a rule in mcstp, $\sum_{k=1}^{m} \varphi_k(M_0, C^G) = m(M_0, C^G)$. Now,*

$$\sum_{i \in N} F_i\left(N_0, C, G\right) = \sum_{k=1}^{m} m(G^k, C^\varphi) + m(M_0, C^G).$$

*By definition of $C^\varphi$, $c_{ij}^\varphi = c_{ij}$ for all $i, j \in G^k$. Thus, $m(G^k, C^\varphi) = m(G^k, C)$. Hence,*

$$\sum_{i \in N} F_i\left(N_0, C, G\right) = \sum_{k=1}^{m} m(G^k, C) + m(M_0, C^G).$$

*By Lemma 3 (i) and (ii), $m(N_0, C) = \sum_{k=1}^{m} m(G^k, C) + m(M_0, C^G)$.*
*Replacing this expression in equation above, we obtain the result.*

*(b) Let $G = \{N\}$. Thus, $F(N_0, C, G) = \varphi(N_0, C^\varphi)$.*
*By definition, $c_{ij}^\varphi = c_{ij}$ for all $i, j \in N$ and $c_{0i}^\varphi = \min_{j \in N} \{c_{0j}\}$ for all $i \in N$. Thus, $C \geq C^\varphi$. Since $\varphi$ satisfies SCM, $\varphi(N_0, C) \geq \varphi(N_0, C^\varphi)$. By Lemma 3 (iii), $m(N_0, C) = m(N_0, C^\varphi)$. Now, $\varphi(N_0, C) = \varphi(N_0, C^\varphi)$.*

*(c) Let $G = \{\{i\}\}_{i \in N}$. For each $i \in N$,*

$$F_i(N_0, C, G) = \varphi_i(\{i\}_0, C^\varphi) = c_{0i}^\varphi = \varphi_i(M_0, C^G)$$

*Moreover, $M_0$ coincides with $N_0$ and $C^G$ coincides with $C$. Therefore, $F(N_0, C, G) = \varphi(N_0, C)$.* ■

We now present the main results of the Section.

**Proposition 5** *$F$ satisfies $RA$, $SYMC$, $SYMA$, $PMC$, and $PMA$.*
***Proof.*** *We divide the proof in several claims.*

***Claim 6*** *$F$ satisfies RA.*
***Proof.*** *Let $(N_0, C, G)$ and $(N_0, C', G)$ be two mcstp with coalition structure satisfying that there exists an mt $t = \{(i^0, i)\}_{i \in N}$ in $(N_0, C, G)$, $(N_0, C', G)$, and $(N_0, C + C', G)$ and an order $\pi = (i_1, \ldots, i_{|N|}) \in \Pi_N$ such that $c_{i_1^0 i_1} \leq c_{i_2^0 i_2} \leq \ldots \leq c_{i_{|N|}^0 i_{|N|}}$ and $c'_{i_1^0 i_1} \leq c'_{i_2^0 i_2} \leq \ldots \leq c'_{i_{|N|}^0 i_{|N|}}$.*
*We first prove that it is possible to find a tree $t$ satisfying the conditions of RA defined above and the three conditions of Lemma 3 for the problems $C$, $C'$, and $C + C'$.*

*Let $t$ be the tree satisfying the conditions of RA. Assume that there exists $G^k \in G$ such that $t_{G^k}$ is not a tree in $G^k$. Since $t$ is a tree in $N_0$, $t_{G^k}$ has no cycles. Let $\{X_1, ..., X_l\}$ the partition of $G^k$ in connected components induced by $t_{G^k}$. Namely, if $i, j \in X_{l'}$ for some $l' = 1, ..., l$, then there is a path in $t_{G^k}$ from $i$ to $j$. Moreover, if $i \in X_{l'}$, $j \in X_{l''}$ and $l' \neq l''$, then there is no path in $t_{G^k}$ from $i$ to $j$.*

*Let $i \in X_{l'}$, $j \in X_{l''}$ and $l' \neq l''$. Since $t$ is a tree there is a path $g_{ij}$ in $t$ from $i$ to $j$. For each arc $(i', j') \in g_{ij}$ we have that $t^1 = (t \setminus \{(i', j')\}) \cup \{(i, j)\}$ is a tree in $(N_0, C)$. Since $t$ is an mt in $(N_0, C)$ and $(N_0, C')$, $c_{ij} \geq c_{i'j'}$ and $c'_{ij} \geq c'_{i'j'}$. Since $i \in X_{l'}$ and $j \in X_{l''}$ we can find $(i', j') \in g_{ij}$ such that $i' \in G^k$, $j' \in G^l$, and $l \neq k$. Because of the definition of $(N_0, C, G)$ and $(N_0, C', G')$ we deduce that $c_{ij} \leq c_{i'j'}$ and $c'_{ij} \leq c'_{i'j'}$. Then, $c_{ij} = c_{i'j'}$ and $c'_{ij} = c'_{i'j'}$. Hence, $t^1$ is an mt in $(N_0, C, G)$, $(N_0, C', G)$, and $(N_0, C + C', G)$.*

*The order $\pi' = (i'_1, \ldots, i'_{|N|}) \in \Pi_N$ obtained by changing in the order $\pi$ the arc $(i', j')$ by $(i, j)$ also satisfies the conditions of the definition of RA.*

*Now, $t^1_{G^k}$ induces a partition of $G^k$ in $l - 1$ connected components. If $l - 1 = 1$, then $t^1_{G^k}$ induces a tree in $G^k$. Otherwise we proceed with $t^1$ as with $t$. Finally, we find an mt $t^{l-1}$ such that $t^{l-1}_{G^k}$ induces a tree in $G^k$.*

*Once we finish with $G^k$ we proceed with the other coalitions. At the end of the procedure we find an mt $t^p$ such that $t^p_{G^k}$ induces a tree in each $G^k \in G$. That is, $t^p$ satisfies the conditions of Lemma 3 (i). Since $t^p$ is a tree in $N_0$, we deduce that $t^p$ also satisfies the conditions of Lemma 3 (ii). Using arguments similar to those used in the proof of Lemma 3 (iii), we can prove that $t^p$ also satisfies the conditions of Lemma 3 (iii).*

*Thus, we can assume that the tree $t$ also satisfies the conditions of Lemma 3.*

*Let $G^k \in G$. Since $t$ satisfies the conditions of Lemma 3 (ii), $\left(M_0, C^G\right)$ and $\left(M_0, C'^G\right)$ are under the conditions of RA. Since $\varphi$ satisfies RA, $\varphi_k\left(M_0, C^G\right) + \varphi_k\left(M_0, C'^G\right) = \varphi_k\left(M_0, C^G + C'^G\right)$. Moreover, it is easy to see that $C^G + C'^G = (C + C')^G$.*

*Since $t$ satisfies the conditions of Lemma 3 (iii), we have that $t^* = t_{G^k} \cup \{(0, i_j)\}$ with $i_j \in G^k$ is an mt in $(G^k_0, C^\varphi)$, $(G^k_0, C'^\varphi)$, and $(G^k_0, (C + C')^\varphi)$.*

*Let $\pi_{G^k} = (i_1, ..., i_{|G^k|})$ be the order induced by $\pi$ over the agents in $G^k$. We have proved above that for all $(j, j') \in t_{G^k}$, $c^\varphi_{0i} \geq c^\varphi_{jj'}$ and $c'^\varphi_{0i} \geq c'^\varphi_{jj'}$. Therefore, $(G^k_0, C^\varphi)$ and $(G^k_0, C'^\varphi)$ are under the conditions of RA. Since $\varphi$ satisfies RA, $\varphi_i(G^k_0, C^\varphi) + \varphi_i(G^k_0, C'^\varphi) = \varphi_i(G^k_0, C^\varphi + C'^\varphi)$ for all $i \in G^k$. Moreover, it is easy to see that $C^\varphi + C'^\varphi = (C + C')^\varphi$.*

*Now, for all $G^k \in G$ and all $i \in G^k$,*

$$\begin{aligned} F_i(N_0, C, G) + F_i(N_0, C', G) &= \varphi_i(G_0^k, C^\varphi) + \varphi_i(G_0^k, C'^\varphi) \\ &= \varphi_i(G_0^k, (C + C')^\varphi) \\ &= F_i(N_0, C + C', G). \end{aligned}$$

■

***Claim 7*** *$F$ satisfies $SYMC$.*

***Proof.*** *Let $G^k$ and $G^{k'}$ be two symmetric coalitions. Then, for all $G^l \in G \backslash \{G^k, G^{k'}\}$,*

$$c_{kl}^G = \min_{i \in G^k, j \in G_0^l} \{c_{ij}\} = \min_{i \in G^{k'}, j \in G_0^l} \{c_{ij}\} = c_{k'l}^G.$$

*That is, $k$ and $k'$ are symmetric agents in $\left(M_0, C^G\right)$. Since $\varphi$ satisfies $SYM$, $\varphi_k\left(M_0, C^G\right) = \varphi_{k'}\left(M_0, C^G\right)$.*

*By Lemma 3 (iii), $m(G_0^k, C^\varphi) = m(G^k, C) + \varphi_k(M_0, C^G)$. Thus,*

$$\begin{aligned} \sum_{i \in G^k} F_i(N_0, C, G) - m(G^k, C) &= \sum_{i \in G^k} \varphi_i(G_0^k, C^\varphi) - m(G^k, C) \\ &= m(G_0^k, C^\varphi) - m(G^k, C) \\ &= \varphi_k(M_0, C^G). \end{aligned}$$

*Repeating the same argument with $G^{k'}$ instead of $G^k$, we obtain that*

$$\sum_{i \in G^{k'}} F_i(N_0, C, G) - m(G^{k'}, C) = \varphi_{k'}(M_0, C^G).$$

*Thus, $F$ satisfies $SYMC$.* ■

***Claim 8*** *$F$ satisfies $SYMA$.*

***Proof.*** *Let $i, j \in G^k \in G$ be symmetric agents in $(N_0, C, G)$. By definition of $C^\varphi$, for all $j' \in G^k$, $c_{ij'}^\varphi = c_{ij'}$ and $c_{jj'}^\varphi = c_{jj'}$. Moreover, $c_{0i}^\varphi = c_{0j}^\varphi = \varphi_k(M_0, C^G)$. Hence, $i$ and $j$ are symmetric agents in $\left(G_0^k, C^\varphi\right)$.*

*Since $\varphi$ satisfies $SYM$, $\varphi_i\left(G_0^k, C^\varphi\right) = \varphi_j\left(G_0^k, C^\varphi\right)$. Thus,*

$$F_i(N_0, C, G) = \varphi_i\left(G_0^k, C^\varphi\right) = \varphi_j\left(G_0^k, C^\varphi\right) = F_j(N_0, C, G).$$

*Hence, $F$ satisfies $SYMA$.* ■

***Claim 9*** *$F$ satisfies $PMC$.*

***Proof.*** *Let $G^k \in G$. Since $\varphi$ satisfies $PM$, $\varphi_l(M_0, C^G) \leq \varphi_l((M\backslash\{k\})_0, C^G)$ for all $l \neq k$.*

*Let $C'^{\varphi}$ denote the matrix $C^{\varphi}$ associated with the problem $(\left(N\backslash G^k\right)_0, C, G\backslash G^k)$.*

*Let $G^l \in G\backslash G^k$. For all $i, j \in G^l$, $c_{ij}^{\varphi} = c_{ij}'^{\varphi}$. For all $i \in G^l$,*

$$c_{0i}^{\varphi} = \varphi_l(M_0, C^G) \leq \varphi_l((M\backslash\{k\})_0, C^G) = c_{0i}'^{\varphi}.$$

*That is $C^{\varphi} \leq C'^{\varphi}$. Let $i \in G^l$. Since $\varphi$ satisfies $SCM$, $\varphi_i(G_0^l, C^{\varphi}) \leq \varphi_i(G_0^l, C'^{\varphi})$. So, $F_i(N_0, C, G) \leq F_i(\left(N\backslash G^k\right)_0, C, G\backslash G^k)$, i.e. $F$ satisfies $PMC$.*
■

***Claim 10*** *$F$ satisfies $PMA$.*

***Proof.*** *Let $G^k \in G$ and $i \in G^k$, $G^k \neq \{i\}$. Let $G' = (G\backslash G^k) \cup (G^k\backslash\{i\})$. By convenience, let us denote as $C'$ the cost matrix $C$ restricted to the problem $((N\backslash\{i\})_0, C, G')$. Notice that $C'$ coincides with $C$ for the agents in $(N\backslash\{i\})_0$.*

*We consider several cases:*

*1. Assume that $c_{kl}^G = c_{kl}'^{G'}$ for all $l \in \{0, 1, ..., m\}$. Thus, $\varphi_l\left(M_0, C^G\right) = \varphi_l\left(M_0, C'^{G'}\right)$ for all $l = 1, ..., m$. Hence, $\left((G^k\backslash\{i\})_0, C^{\varphi}\right) = \left((G^k\backslash\{i\})_0, C'^{\varphi}\right)$.*

- *Since $\varphi$ satisfies $PM$, $\varphi_j\left(G_0^k, C^{\varphi}\right) \leq \varphi_j\left((G^k\backslash\{i\})_0, C^{\varphi}\right)$ for all $j \in G^k\backslash\{i\}$. Then,*

$$\begin{aligned} F_j(N_0, C, G) &= \varphi_j\left(G_0^k, C^{\varphi}\right) \leq \varphi_j\left((G^k\backslash\{i\})_0, C^{\varphi}\right) \\ &= \varphi_j\left((G^k\backslash\{i\})_0, C'^{\varphi}\right) \\ &= F_j((N\backslash\{i\})_0, C, G') \end{aligned}$$

*for all $j \in G^k\backslash\{i\}$.*

- *Let $G^l \in G$ such that $l \neq k$. Then, $c_{jj'}^{\varphi} = c_{jj'}'^{\varphi}$ for all $j, j' \in G^l \cup \{0\}$. Hence, $\varphi_j(G_0^l, C^{\varphi}) = \varphi_j(G_0^l, C'^{\varphi})$ for all $j \in G^l$. So,*

$$\begin{aligned} F_j(N_0, C, G) &= \varphi_j(G_0^l, C^{\varphi}) = \varphi_j(G_0^l, C'^{\varphi}) \\ &= F_j((N\backslash\{i\})_0, C', G') \end{aligned}$$

*for all $j \in G^l$.*

*2. Assume that $c_{kk^*}^G \neq c_{kk^*}'^{G'}$ for some $k^* \in \{0, 1, ..., m\}$. Then, $c_{kk^*}^G < c_{kk^*}'^{G'}$. Moreover, $c_{ll^*}^G \leq c_{ll^*}'^{G'}$ for all $l, l^* \in \{0, 1, ..., m\}$.*

*Since $\varphi$ satisfies $SCM$, $\varphi_k\left(M_0, C^G\right) \leq \varphi_k\left(M_0, C'^{G'}\right)$. Now, $c_{jj^*}^{\varphi} = c_{jj^*}'^{\varphi}$ for all $j, j^* \in G^k\backslash\{i\}$ and $c_{0j}^{\varphi} \leq c_{0j}'^{\varphi}$ for all $j \in G^k\backslash\{i\}$. Since $\varphi$ satisfies $SCM$, $\varphi_j\left((G^k\backslash\{i\})_0, C^{\varphi}\right) \leq \varphi_j\left((G^k\backslash\{i\})_0, C'^{\varphi}\right)$ for all $j \in G^k\backslash\{i\}$.*

*Since* $\varphi$ *satisfies* $PM$, *we have that* $\varphi_j\left(G_0^k, C^{\varphi}\right) \leq \varphi_j\left((G^k\backslash\{i\})_0, C^{\varphi}\right).$ *Then,*

$$\begin{aligned} F_j(N_0, C, G) &= \varphi_j\left(G_0^k, C^{\varphi}\right) \leq \varphi_j\left((G^k\backslash\{i\})_0, C^{\varphi}\right) \\ &\leq \varphi_j\left((G^k\backslash\{i\})_0, C'^{\varphi}\right) \\ &= F_j((N\backslash\{i\})_0, C', G') \end{aligned}$$

*for all* $j \in G^k\backslash\{i\}$. ■

■

**Proposition 11** *There is a unique rule satisfying* $RA$, $SYMC$, $SYMA$, $PMC$, *and* $PMA$.

***Proof.*** *Let* $f$ *be a rule in mcstp with coalition structure satisfying* $RA$, $SYMC$, $SYMA$, $PMC$, $PMA$. *We prove that* $f = F$. *We proceed with several claims.*

***Claim 12*** *If* $G = \{\{i\}\}_{i\in N}$ *or* $G = \{N\}$, *then* $f(N_0, C, G) = \varphi(N_0, C)$.

***Proof.*** *Let* $G = \{\{i\}\}_{i\in N}$, *i.e. each agent forms a coalition. Given an mcstp* $(N_0, C)$ *we define* $f'(N_0, C) = f(N_0, C, G)$. *Then,*

$$\sum_{i\in N} f_i'(N_0, C) = \sum_{i\in N} f_i(N_0, C, G) = m(N_0, C).$$

*Hence,* $f'$ *is a rule in mcstp.*

*Since* $f$ *satisfies* $SYMC$ *in mcstp with coalition structure,* $f'$ *satisfies* $SYM$ *in mcstp. Since* $f$ *satisfies* $PMC$ *in mcstp with coalition structure,* $f'$ *satisfies* $PM$ *in mcstp. Moreover,* $f'$ *also satisfies* $RA$. *By Lemma 1* $(e)$, $\varphi$ *is the unique rule in mcstp satisfying* $SYM$, $RA$, *and* $PM$. *Thus,*

$$f(N_0, C, G) = f'(N_0, C) = \varphi(N_0, C).$$

*Let* $G = \{N\}$, *i.e. all agents are in the same coalition. Given an mcstp* $(N_0, C)$ *we define* $f'(N_0, C) = f(N_0, C, G)$. *As above,* $f'$ *is a rule in mcstp.*

*Since* $f$ *satisfies* $SYMA$ *in mcstp with coalition structure,* $f'$ *satisfies* $SYM$ *in mcstp. Since* $f$ *satisfies* $PMA$ *in mcstp with coalition structure,* $f'$ *satisfies* $PM$ *in mcstp. Moreover,* $f'$ *also satisfies* $RA$. *By Lemma 1* $(e)$, $\varphi$ *is the unique rule in mcstp satisfying* $SYM$, $RA$, *and* $PM$. *Thus,*

$$f(N_0, C, G) = f'(N_0, C) = \varphi(N_0, C).$$

■

***Claim 13*** *For each mcstp with coalition structure* $(N_0, C, G)$ *and each* $G^k \in G$, *let* $(N_0', C', G')$ *be the problem obtained from* $(N_0, C, G)$ *by considering that the rest of the coalitions have a unique agent whose connection cost to the rest of the agents is given by* $\left(M_0, C^G\right)$. *Namely,* $N' = G^k \cup \left( \bigcup_{l \in M \backslash \{k\}} \{i_l\} \right)$, $G' = \{G^k\} \cup \{\{i_l\}\}_{l \in M \backslash \{k\}}$, *and* $C'$ *is defined as follows: if* $i, j \in G^k \cup \{0\}$, *then* $c'_{ij} = c_{ij}$. *If* $i \in G^k$ *and* $j = i_l$ *with* $l \in M \backslash \{k\}$, *then* $c'_{ij} = c^G_{kl}$. *If* $i = 0$ *and* $j = i_l$ *with* $l \in M \backslash \{k\}$, *then* $c'_{ij} = c^G_{0l}$. *If* $i = i_l$, $j = i_{l'}$ *and* $k \notin \{l, l'\}$, *then* $c'_{ij} = c^G_{ll'}$.

*Then, for each* $i \in G^k$,

$$f_i(N_0, C, G) = f_i(N_0', C', G').$$

***Proof.*** *Let* $(N_0, C, G)$ *be an mcstp with coalition structure and* $G^k \in G$. *We assume, wlog, that* $k = m$.

*We take* $(N_0'^0, C'^0, G'^0) = (N_0, C, G)$. *For each* $l = 1, ..., m-1$ *we define* $(N_0'^l, C'^l, G'^l)$ *as follows.*

- $N'^l = N'^{l-1} \cup \{i_l\}$.
- $\left(G'^l\right)^l = \left(G'^{l-1}\right)^l \cup \{i_l\} = G^l \cup \{i_l\}$. *For any* $l' \neq l$, $\left(G'^l\right)^{l'} = \left(G'^{l-1}\right)^{l'}$.
- $C'^l$ *is defined as follows:*

$$c'^l_{ij} = \begin{cases} c_{ij} & \text{if } i, j \in N_0'^{l-1} \\ 0 & \text{if } i = i_l,\ j \in G^l \\ c^G_{ll'} & \text{if } i = i_l,\ j \in G'^{l'},\ l \neq l' \end{cases}$$

*For each* $l = 1, ..., m-1$ *and for each* $l' \neq l$,

$$\min_{i \in \left(G'^l\right)^l, j \in \left(G'^l\right)^{l'}} \{c'^l_{ij}\} = \min_{i \in \left(G'^{l-1}\right)^l, j \in \left(G'^{l-1}\right)^{l'}} \{c'^{l-1}_{ij}\}.$$

*Since* $f$ *satisfies* $PMA$, *for all* $i \in G^m$ *and all* $l = 1, ..., m-1$,

$$f_i(N_0'^{l-1}, C'^{l-1}, G'^{l-1}) = f_i(N_0'^l, C'^l, G'^l).$$

*Now,*

$$f_i(N_0, C, G) = f_i(N_0'^0, C'^0, G'^0) = f_i(N_0'^{m-1}, C'^{m-1}, G'^{m-1}).$$

*Since* $f$ *satisfies* $PMA$, *for all* $i \in G^k$,

$$f_i(N_0'^{m-1}, C'^{m-1}, G'^{m-1}) = f_i(N_0', C', G').$$

■

***Claim 14*** *For each mcstp with coalition structure* $(N_0, C, G)$ *and each* $G^k \in G$,

$$\sum_{i \in G^k} f_i(N_0, C, G) = f_k\left(M_0, C^G, \{\{l\}\}_{l=1}^m\right) + m\left(G^k, C\right).$$

***Proof.*** *Consider the problem among coalitions* $\left(M_0', C'^{G'}\right)$ *associated with* $(N_0', C', G')$ *defined in Claim 13. It is trivial to see that* $C'^{G'}$ *coincides with* $C^G$.

*Applying Lemma 3 it is easy to deduce*

$$m(N_0', C') = m\left(M_0, C^G\right) + m(G^k, C).$$

*By Claim 13,* $\sum_{i \in G^k} f_i(N_0, C, G) = \sum_{i \in G^k} f_i(N_0', C', G')$. *Then,*

$$\begin{aligned}
\sum_{l \in M \setminus \{k\}} f_{i_l}(N_0', C', G') + \sum_{i \in G^k} f_i(N_0, C, G) &= \sum_{l \in M \setminus \{k\}} f_{i_l}(N_0', C', G') + \sum_{i \in G^k} f_i(N_0', C', G') \\
&= m\left(M_0, C^G\right) + m(G^k, C) \\
&= \sum_{l=1}^{m} f_l\left(M_0, C^G, \{\{l\}\}_{l=1}^m\right) + m(G^k, C).
\end{aligned}$$

*Now, it is enough to prove that for all* $l \in M \setminus \{k\}$,

$$f_{i_l}(N_0', C', G') = f_l\left(M_0, C^G, \{\{l\}\}_{l=1}^m\right).$$

*Let* $l \in M \setminus \{k\}$. *Applying Claim 13 to* $(N_0', C', G')$ *with* $G'^l$ *instead of* $G^k$ *we obtain that*

$$f_{i_l}(N_0', C', G') = f_{i_l}(N_0'', C'', G'')$$

*where* $N'' = \{i_1, ..., i_m\}$, $c''_{i_j i_{j'}} = c^G_{jj'}$ *for all* $j, j' = 0, 1, ..., m$, *and* $G'' = \{\{i_j\}\}_{j=1}^m$. *Notice that* $(N_0'', C'', G'')$ *is equivalent to* $\left(M_0, C^G, \{\{l\}\}_{l=1}^m\right)$. *By Claim 12 we have*

$$f_{i_l}(N_0'', C'', G'') = \varphi_{i_l}(N_0'', C'') = \varphi_l\left(M_0, C^G\right) = f_l\left(M_0, C^G, \{\{l\}\}_{l=1}^m\right).$$

■

***Claim 15*** *It is enough to prove that* $f$ *is unique on the subclass of mcstp* $(N_0, C, G)$ *satisfying that there exists* $x \in \mathbb{R}_+$ *and a network* $g$ *such that* $c_{ij} = x$ *if* $(i,j) \in g$ *and* $c_{ij} = 0$ *otherwise.*

***Proof.*** *Norde et al. (2004) proved that if* $C$ *is a cost matrix, then there exists a family* $\{C^p\}_{p=1}^a$ *of cost matrices satisfying three conditions:*

*1.* $C = \sum_{p=1}^a C^p$.

*2. For each $p \in \{1,...,a\}$ there exist $x^p \in \mathbb{R}$ and a network $g^p$ such that $c_{ij}^p = x_p$ if $(i,j) \in g^p$ and $c_{ij}^p = 0$ otherwise.*

*3. There exists $\sigma : \{(i,j)\}_{i,j \in N_0, i<j} \to \left\{1,2,...,\frac{n(n+1)}{2}\right\}$ such that if $i,j,k,l \in N$ with $i<j$, $k<l$, and $\sigma(i,j) < \sigma(k,l)$, then $c_{ij} \leq c_{kl}$ and $c_{ij}^p \leq c_{kl}^p$ for all $p \in \{1,...,a\}$.*

*By condition 3, $C^1$ and $\sum_{p=2}^a C^p$ satisfy the conditions of the definition of RA. Hence,*

$$f(N_0,C,G) = f\left(N_0,C^1,G\right) + f\left(N_0, \sum_{p=2}^a C^p, G\right).$$

*By condition 3, $C^2$ and $\sum_{p=3}^a C^p$ satisfy the conditions of the definition of RA. Hence,*

$$f\left(N_0, \sum_{p=2}^a C^p, G\right) = f\left(N_0,C^2,G\right) + f\left(N_0, \sum_{p=3}^a C^p, G\right).$$

*Repeating the same argument we obtain that*

$$f(N_0,C,G) = \sum_{p=1}^a f(N_0,C^p,G).$$

*By condition 2, Claim 15 holds.* ■

*Let $(N_0,C,G)$ be an mcstp with coalition structure and $G^k \in G$. By Claim 13 we can assume that $(N_0,C,G)$ has the same structure as the problem $(N_0',C',G')$ defined in Claim 13.*

*By Claim 15 we can assume that there exists $x \in \mathbb{R}_+$ and a network $g$ such that $c_{ij} = x$ if $(i,j) \in g$ and $c_{ij} = 0$ otherwise.*

***Claim 16*** *Let $G^k \in G$. Assume that there exists $l \in M_0 \backslash \{k\}$, and $i' \in G^k$ such that $c_{i'i_l} = 0$. Then, for each $i \in G^k$,*

$$f_i(N_0,C,G) = \frac{f_k\left(M_0,C^G,\{\{l\}\}_{l=1}^m\right)}{|G^k|}.$$

***Proof.*** *We know that $\max_{i,j\in G^k}\{c_{ij}\} \leq \min_{i\in G^k, j\in N_0\backslash G^k}\{c_{ij}\}$. Since $\min_{i\in G^k, j\in N_0\backslash G^k}\{c_{ij}\} \leq c_{i'i_l} = 0$, we have that $c_{ij} = 0$ for all $i,j \in G^k$. Therefore, $m(G^k,C) = 0$. By Claim 14,*

$$\sum_{i\in G^k} f_i(N_0,C,G) = f_k\left(M_0,C^G,\{\{l\}\}_{l=1}^m\right).$$

*Moreover, $c_{0i} \in \{0, x\}$ for all $i \in G^k$. Two cases are possible:*

*1. There exist $i^* \in G^k$ such that $c_{0i^*} = 0$. Then, $c_{0k}^G = \min\limits_{i \in G^k} \{c_{0i}\} = 0$.*

*By Lemma 3 $(b)$, for all $S \subset G$, $v_{(C^G)^*}(S \cup \{k\}) - v_{(C^G)^*}(S) = 0$. Thus, $\varphi_k(M_0, C^G) = Sh_k(M, v_{(C^G)^*}) = 0$. By Claim 12, $f_k(M_0, C^G, \{\{l\}\}_{l=0}^m) = \varphi_k(M_0, C^G)$. Thus,*

$$f_k\left(M_0, C^G, \{\{l\}\}_{l=0}^m\right) = 0.$$

*Let $i \in G^k$. By $PMG$, $f_i(N_0, C, G) \leq f_i(G_0^k, C, \{G^k\})$. By Claim 12, $f_i(G_0^k, C, \{G^k\}) = \varphi_i(G_0^k, C)$. Since $c_{jj'} = 0$ for all $j, j' \in G_0^k$, $\varphi_i(G_0^k, C) = 0$. Thus,*

$$f_i(N_0, C, G) = 0.$$

*2. $c_{0i} = x$ for all $i \in G^k$.*

*We first prove that given $i, i' \in G^k$, $i$ and $i'$ are symmetric agents. We know $c_{0i} = c_{0i'} = x$. We have seen that $c_{ij} = c_{i'j} = 0$ for all $j \in G^k \backslash \{i, i'\}$. Since $(N_0, C, G)$ has the same structure as the problem $(N_0', C', G')$ defined in Claim 13, $c_{ij} = c_{i'j}$ for all $j \in N \backslash G^k$.*

*Since $f$ satisfies $SYMA$, $f_i(N_0, C, G) = f_{i'}(N_0, C, G)$ for all $i, i' \in G^k$. Thus, for all $i \in G^k$,*

$$f_i(N_0, C, G) = \frac{\sum_{i \in G^k} f_i(N_0, C, G)}{|G^k|} = \frac{f_k\left(M_0, C^G, \{\{l\}\}_{l=1}^m\right)}{|G^k|}.$$

■

***Claim 17*** *Assume that for all $l \in M_0 \backslash \{k\}$ and all $i \in G^k$, $c_{ii_l} = x$. Thus, for each $i \in G^k$,*

$$f_i(N_0, C, G) = f_i\left(G_0^k, C, \{G^k\}\right).$$

***Proof.*** *Let $t'$ be an mt in $(G_0^k, C)$ and let $t''$ be an mt in $((N \backslash G^k)_0, C)$. Following Prim's algorithm, we can construct an mt $t$ in $(N_0, C, G)$ such that $t = t' \cup t''$. Therefore,*

$$m(N_0, C) = m\left(G_0^k, C\right) + m\left(\left(N \backslash G^k\right)_0, C\right).$$

*Since $f$ satisfies $PMC$, $f_i(N_0, C, G) \leq f_i(G_0^k, C, \{G^k\})$ for all $i \in G^k$ and $f_i(N_0, C, G) \leq f_i\left(\left(N \backslash G^k\right)_0, C, \{\{G^l\}\}_{l \in M \backslash \{k\}}\right)$ for all $i \in N \backslash G^k$. Thus,*

$$\begin{aligned}
m(N_0, C) &= \sum_{i \in G^k} f_i(N_0, C, G) + \sum_{i \in N \backslash G^k} f_i(N_0, C, G) \\
&\leq \sum_{i \in G^k} f_i\left(G_0^k, C, \{G^k\}\right) + \sum_{i \in N \backslash G^k} f_i\left(\left(N \backslash G^k\right)_0, C, \{\{G^l\}\}_{l \in M \backslash \{k\}}\right) \\
&= m\left(G_0^k, C\right) + m\left(\left(N \backslash G^k\right)_0, C\right).
\end{aligned}$$

*Thus, $f_i(N_0, C, G) = f_i(G_0^k, C, \{G^k\})$ for all $i \in G^k$.* ■

***Claim 18*** *For all $i \in G^k$, $f_i(N_0, C, G) = F_i(N_0, C, G)$.*

***Proof.*** *We distinguish two cases, given by Claims 16 and 17.*

*1. There exists $l \in M_0 \backslash \{k\}$, and $i' \in G^k$ such that $c_{i' i_l} = 0$.*

*Let $i \in G^k$. By Claim 16,*

$$f_i(N_0, C, G) = \frac{f_k\left(M_0, C^G, \{\{l\}\}_{l=0}^m\right)}{|G^k|}.$$

*By Claim 12, $f_k\left(M_0, C^G, \{\{l\}\}_{l=0}^m\right) = \varphi_k(M_0, C^G)$. So,*

$$f_i(N_0, C, G) = \frac{\varphi_k(M_0, C^G)}{|G^k|}.$$

*Consider now the problem $(G_0^k, C^\varphi)$ where $c_{jj'}^\varphi = c_{jj'}$ if $0 \notin \{j, j'\}$ and $c_{0j}^\varphi = \varphi_k(M_0, C^G)$ for all $j \in G^k$.*

*We have seen in the proof of Claim 16 that $c_{jj'} = 0$ for all $j, j' \in G^k$. Therefore, $m(G_0^k, C^\varphi) = \varphi_k(M_0, C^G)$ and all agents in $G^k$ are symmetric in $(G_0^k, C^\varphi)$. Since $\varphi$ satisfies $SYM$,*

$$\varphi_i(G_0^k, C^\varphi) = \frac{\varphi_k(M_0, C^G)}{|G^k|}.$$

*Then,*

$$f_i(N_0, C, G) = \varphi_i(G_0^k, C^\varphi) = F_i(N_0, C, G).$$

*2. Assume that for all $l \in M_0 \backslash \{k\}$ and all $i \in G^k$, $c_{ii_l} = x$.*

*Let $i \in G^k$. By Claim 17, $f_i(N_0, C, G) = f_i\left(G_0^k, C, \left\{G^k\right\}\right)$. By Claim 12, $f_i\left(G_0^k, C, \left\{G^k\right\}\right) = \varphi_i\left(G_0^k, C\right)$. Thus,*

$$f_i(N_0, C, G) = \varphi_i\left(G_0^k, C\right).$$

*Consider now the problem $(G_0^k, C^\varphi)$. We know that $c_{jj'}^\varphi = c_{jj'}$ if $0 \notin \{j, j'\}$ and $c_{0j}^\varphi = \varphi_k(M_0, C^G)$ for all $j \in G^k$.*

*For all $l \neq k$, $c_{kl}^G = x$. Now it is not difficult to prove that for all $S \subset M_0$, $v_{C^{G*}}(S \cup \{k\}) - v_{C^{G*}}(S) = x$. Thus, $\varphi_k\left(M_0, C^G\right) = Sh_k\left(M, v_{C^{G*}}\right) = x$.*

*Hence, $(G_0^k, C^\varphi) = \left(G_0^k, C\right)$. Then,*

$$f_i(N_0, C, G) = \varphi_i\left(G_0^k, C\right) = \varphi_i(G_0^k, C^\varphi) = F_i(N_0, C, G).$$

■

*And the proof of Proposition 11 is completed.* ■

The next theorem is a trivial consequence of Propositions 5 and 11.

**Theorem 19** *$F$ is the unique rule satisfying $RA$, $SYMC$, $SYMA$, $PMC$, and $PMA$.*

**Proposition 20** *The properties used in Theorem 19 are independent.*

***Proof.*** *We prove that if we remove some of the properties of Theorem 19, we can find more rules satisfying the other properties. We do it by considering several claims. In each claim we define a rule satisfying four properties but failing the other. We do not make the proofs rigorously in order not to enlarge the paper. We simply give an idea of the proof.*

***Claim 21*** *There exist rules satisfying $RA$, $SYMC$, $SYMA$, and $PMC$ but failing $PMA$.*

***Proof.*** *We define the rule $f^1$ as follows. Let $(N_0, C, G)$ be an mcstp with coalition structure and $i \in G^k \in G$. Then,*

$$f_i^1(N_0, C, G) = \frac{\varphi_k\left(M_0, C^G\right) + m(G^k, C)}{|G^k|}.$$

*1. $f^1$ satisfies $RA$. Using arguments similar to those used in the proof of Claim 6 we can prove that*

$$\varphi_k\left(M_0, (C + C')^G\right) = \varphi_k\left(M_0, C^G\right) + \varphi_k\left(M_0, C'^G\right)$$

*and*

$$m(G^k, C + C') = m(G^k, C) + m(G^k, C').$$

*Now it is trivial to see that*

$$f_i^1(N_0, C + C', G) = f_i^1(N_0, C, G) + f_i^1(N_0, C', G).$$

*2. $f^1$ satisfies $SYMC$. Let $G^k$ and $G^{k'}$ be two symmetric coalitions. Then, $k$ and $k'$ are symmetric agents in $\left(M_0, C^G\right)$. Since $\varphi$ satisfies $SYM$, $\varphi_k\left(M_0, C^G\right) = \varphi_{k'}\left(M_0, C^G\right)$. Now,*

$$\begin{aligned}
\sum_{i \in G^k} f_i^1(N_0, C, G) - m(G^k, C) &= \varphi_k\left(M_0, C^G\right) \\
&= \varphi_{k'}\left(M_0, C^G\right) \\
&= \sum_{i \in G^{k'}} f_i^1(N_0, C, G) - m(G^{k'}, C).
\end{aligned}$$

*3. $f^1$ satisfies $SYMA$. It is trivial.*

*4. $f^1$ satisfies $PMC$. Let $G^k \in G$. Since $\varphi$ satisfies $PM$, $\varphi_l(M_0, C^G) \leq \varphi_l((M\backslash\{k\})_0, C^G)$ for all $l \neq k$.*

*Thus, for all $i \in G^l$, $l \neq k$,*

$$\begin{aligned} f_i^1(N_0, C, G) &= \frac{\varphi_l\left(M_0, C^G\right) + m(G^l, C)}{|G^l|} \\ &\leq \frac{\varphi_l((M\backslash\{k\})_0, C^G) + m(G^l, C)}{|G^l|} \\ &= f_i^1(\left(N\backslash G^k\right)_0, C, G\backslash G^k). \end{aligned}$$

*5. $f^1$ fails $PMA$. Assume that $G = \{N\}$. Thus, $f^1$ divides $m\left(N_0, C\right)$ equally among the agents. In this case it is trivial to see that $f^1$ does not satisfy $PMA$.* ■

***Claim 22*** *There exist rules satisfying $RA$, $SYMC$, $SYMA$, and $PMA$ but failing $PMC$.*

***Proof.*** *We define the rule $f^2$ as follows. Let $(N_0, C, G)$ be an mcstp with coalition structure and $i \in G^k$. Thus,*

$$f_i^2\left(N_0, C, G\right) = Sh_i\left(G^k, v_C^0\right) + \frac{m\left(M_0, C^G\right)}{|M|\,|G^k|}$$

*where for all $S \subset G^k$, $v_C^0\left(S\right) = m\left(S, C_{|S}^*\right)$ and $C_{|S}^*$ is the irreducible matrix associated with the problem $\left(S, C\right)$.*

*1. $f^2$ satisfies $RA$. By Lemma 1 $(d)$, for all $S \subset N$, $v_{(C+C')^*}\left(S\right) = v_{C^*}\left(S\right) + v_{C'^*}\left(S\right)$. Using similar arguments we can conclude that for all $S \subset G^k$, $v_{C+C'}^0\left(S\right) = v_C^0\left(S\right) + v_{C'}^0\left(S\right)$. Since $Sh$ is additive on $v$, we conclude that*

$$Sh_i\left(G^k, v_{C+C'}^0\right) = Sh_i\left(G^k, v_C^0\right) + Sh_i\left(G^k, v_{C'}^0\right).$$

*We have proved in the proof of Claim 6 that $C^G$ and $C'^G$ are also under the conditions of $RA$. Thus,*

$$m\left(M_0, \left(C + C'\right)^G\right) = m\left(M_0, C^G\right) + m\left(M_0, C'^G\right).$$

*Now, it is obvious that $f^2$ satisfies $RA$.*

*2. $f^2$ satisfies SYMC. Let $G^k$ and $G^{k'}$ be two symmetric coalitions.*

$$\begin{aligned}
\sum_{i\in G^k} f_i^2(N_0,C,G)-m(G^k,C) &= \sum_{i\in G^k} Sh_i\left(G^k,v_C^0\right)+\frac{m\left(M_0,C^G\right)}{|M|}-m(G^k,C)\\
&= v_C^0\left(G^k\right)+\frac{m\left(M_0,C^G\right)}{|M|}-m(G^k,C)\\
&= m(G^k,C^*_{|G^k})+\frac{m\left(M_0,C^G\right)}{|M|}-m(G^k,C)\\
&= m(G^k,C)+\frac{m\left(M_0,C^G\right)}{|M|}-m(G^k,C)\\
&= \frac{m\left(M_0,C^G\right)}{|M|}.
\end{aligned}$$

*Analogously,*

$$\sum_{i\in G^k} f_i^2(N_0,C,G)-m(G^k,C)=\frac{m\left(M_0,C^G\right)}{|M|}.$$

*Hence, $f^2$ satisfies SYMC.*

*3. $f^2$ satisfies SYMA. Let $i,j\in G^k$ be a pair of symmetric agents. It is trivial to see that $i$ and $j$ are also symmetric in $\left(G^k,v_C^0\right)$. Then, $Sh_i\left(G^k,v_C^0\right)=Sh_j\left(G^k,v_C^0\right)$. Hence, $f_i^2(N_0,C,G)=f_j^2(N_0,C,G)$.*

*4. $f^2$ satisfies PMA. Let $i\in G^k$ be such that $G^k\setminus\{i\}\neq\varnothing$.*

*We know that for all $j\in G^k\setminus\{i\}$,*

$$Sh_j\left(G^k,v_C^0\right)=\frac{1}{|G^k|!}\sum_{\pi\in\Pi_{G^k}}\left[v_C^0\left(Pre\left(j,\pi\right)\cup\{j\}\right)-v_C^0\left(Pre\left(j,\pi\right)\right)\right].$$

*By Lemma 1 (c), $v_{C^*}$ is a concave game. Using similar arguments we can prove that $v_C^0$ is a concave game. Then, for all $\pi\in\Pi_{G^k}$,*

$$v_C^0\left(Pre\left(j,\pi\right)\cup\{j\}\right)-v_C^0\left(Pre\left(j,\pi\right)\right)\le v_C^0\left(\left(Pre\left(j,\pi\right)\setminus\{i\}\right)\cup\{j\}\right)-v_C^0\left(\left(Pre\left(j,\pi\right)\setminus\{i\}\right)\right).$$

*Making some computations it is possible to prove that for all $j\in G^k\setminus\{i\}$,*

$$Sh_j\left(G^k,v_C^0\right)\le Sh_j\left(G^k\setminus\{i\},v_C^0\right).$$

*Let us denote $(N^{-i},C,G^{-i})=\left((N\setminus\{i\})_0,C,\left(G\setminus G^k\right)\cup\left(G^k\setminus\{i\}\right)\right)$. Then,*

$m\left(M_0, C^{G_{-i}}\right) \geq m\left(M_0, C^G\right)$. *Now, for all* $j \in G^k \backslash \{i\}$,

$$\begin{aligned} f_j^2(N_0, C, G) &= Sh_j\left(G^k, v_C^0\right) + \frac{m\left(M_0, C^G\right)}{|M|\,|G^k|} \\ &\leq Sh_j\left(G^k \backslash \{i\}, v_C^0\right) + \frac{m\left(M_0, C^{G_{-i}}\right)}{|M|\left(|G^k|-1\right)} \\ &= f_j^2\left(N^{-i}, C, G^{-i}\right). \end{aligned}$$

*Assume that for each* $G^l$ *with* $l \neq k$, $\min\limits_{j \in G^k, j' \in G^l} \{c_{jj'}\} = \min\limits_{j \in G^k \backslash \{i\}, j' \in G^l} \{c_{jj'}\}$. *Let* $j \in G^l \in G \backslash G^k$. *Then*

$$\begin{aligned} f_j^2(N_0, C, G) &= Sh_j\left(G^l, v_C^0\right) + \frac{m\left(M_0, C^{G_{-i}}\right)}{|M|\,|G^l|} \\ &= Sh_j\left(G^l, v_C^0\right) + \frac{m\left(M_0, C^G\right)}{|M|\,|G^l|}. \end{aligned}$$

*5.* $f^2$ *fails* $PMC$. *Assume that* $G = \{\{i\}\}_{i \in N}$. *Thus,* $f^2$ *divides* $m(N_0, C)$ *equally among the agents. Now it is trivial to see that* $f^2$ *does not satisfy* $PMC$. ■

***Claim 23*** *There exist rules satisfying* $RA$, $SYMC$, $PMC$, *and* $PMA$ *but failing* $SYMA$.

***Proof.*** *We define the rule* $f^3$ *as follows. Given* $T \subset \mathcal{N}$ *finite, let* $\pi^N$ *denote the order induced in* $N$ *by the index of the agents. Namely, given* $i, j \in N$, $\pi^N(i) < \pi^N(j)$ *if and only if* $i < j$. *For each mcstp* $(N_0, C)$ *and* $i \in N$ *we define*

$$\psi_i(N_0, C) = v_{C^*}\left(Pre\left(i, \pi^N\right) \cup \{i\}\right) - v_{C^*}\left(Pre\left(i, \pi^N\right)\right).$$

*Let* $(N_0, C, G)$ *be an mcstp with coalition structure and* $i \in G^k$. *Thus,*

$$f_i^3(N_0, C, G) = \psi_i\left(G_0^k, C^\varphi\right).$$

*1.* $f^3$ *satisfies* $RA$. *Let* $C$ *and* $C'$ *as in the definition of* $RA$. *Proceeding as in the proof of Claim 6, we obtain that* $(G_0^k, C^\varphi)$ *and* $(G_0^k, C'^\varphi)$ *are under the conditions of* $RA$. *Bergantiños and Vidal-Puga (2007c) proved that* $\psi$ *satisfies* $RA$. *Therefore,* $\psi_i(G_0^k, (C+C')^\varphi) = \psi_i(G_0^k, C^\varphi) + \psi_i(G_0^k, C'^\varphi)$ *for all* $i \in G^k$. *Thus, given* $i \in G^k \in G$,

$$\begin{aligned} f_i^3(N_0, C + C', G) &= \psi_i(G_0^k, (C+C')^\varphi) \\ &= \psi_i(G_0^k, C^\varphi) + \psi_i(G_0^k, C'^\varphi) \\ &= f_i^3(N_0, C, G) + f_i^3(N_0, C', G). \end{aligned}$$

*2. $f^3$ satisfies SYMC. Let $G^k$ and $G^{k'}$ be two symmetric coalitions. Then, $k$ and $k'$ are symmetric agents in $\left(M_0, C^G\right)$. Since $\varphi$ satisfies SYM, $\varphi_k\left(M_0, C^G\right) = \varphi_{k'}\left(M_0, C^G\right)$.*

*By Lemma 3 (iii), $m(G_0^k, C^\varphi) = \varphi_k(M_0, C^G) + m(G^k, C)$.*

*Therefore,*

$$\begin{aligned}\sum_{i \in G^k} f_i^3(N_0, C, G) - m(G^k, C) &= \sum_{i \in G^k} \psi_i(G_0^k, C^\varphi) - m(G^k, C) \\ &= m(G_0^k, C^\varphi) - m(G^k, C) \\ &= \varphi_k(M_0, C^G)\end{aligned}$$

*Proceeding in the same way for $G^{k'}$ we obtain that*

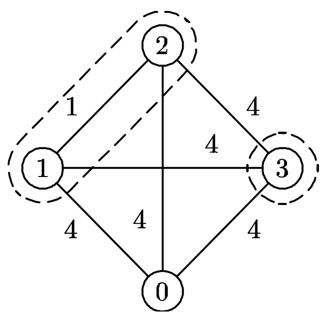
$$\sum_{i \in G^{k'}} f_i^3(N_0, C, G) - m(G^{k'}, C) = \varphi_{k'}(M_0, C^G).$$

*Therefore, $f^3$ satisfies SYMC.*

*3. $f^3$ satisfies PMC. It is not difficult to prove that $\psi$ satisfies SCM. Using arguments similar to those used in the proof of Claim 9, we can prove that $f^3$ satisfies PMC.*

*4. $f^3$ satisfies PMA. It is not difficult to prove that $\psi$ satisfies PM. Using arguments similar to those used in the proof of Claim 10, we can prove that $f^3$ satisfies PMA.*

*5. $f^3$ fails SYMA. Consider the mcstp with coalition structure where $N = \{1, 2, 3\}$, $G = \{G^1, G^2\}$, $G^1 = \{1, 2\}$, $G^2 = \{3\}$ and matrix $C$ which is represented in the following figure:*




*Agents 1 and 2 are symmetric. $\varphi_1(M_0, C^G) = \varphi_2(M_0, C^G) = 4$. Therefore, $c_{01}^\varphi = c_{02}^\varphi = 4$ and $c_{23}^\varphi = 1$. Now, $f_1^3(N_0, C, G) = 4$ and $f_2^3(N_0, C, G) = 1$.* ■

***Claim 24*** *There exist rules satisfying RA, SYMA, PMC, and PMA but failing SYMC.*

***Proof.*** *We define the rule $f^4$ as follows. Let $(N_0, C, G)$ be an mcstp with coalition structure and $i \in G^k \in G$.*

*Let $\pi'$ be an order over the set of all finite subsets of $\mathcal{N}$, $\pi'$ induces an order over the elements of $G$. We also denote this order as $\pi'$. We define the rule $\phi$ over $\left(M_0, C^G\right)$. For each $G^l \in G$,*

$$\phi_l\left(M_0, C^G\right) = v_{(C^G)^*}\left(Pre\left(l, \pi'\right) \cup \{l\}\right) - v_{(C^G)^*}\left(Pre\left(l, \pi'\right)\right).$$

*Now,*

$$f_i^4\left(N_0, C, G\right) = \varphi_i\left(G_0^k, C^\phi\right)$$

*where*

$$c^\phi_{jj'} = \begin{cases} c_{jj'} & \text{if } 0 \notin \{j, j'\} \\ \phi_k\left(M_0, C^G\right) & \text{if } 0 \in \{j, j'\}. \end{cases}$$

*1. $f^4$ satisfies RA.*

*We have proved in Claim 6 that $\left(M_0, C^G\right)$ and $\left(M_0, C'^G\right)$ are under the conditions of RA. Moreover, $\left(C + C'\right)^G = C^G + C'^G$.*

*By Lemma 1 $(d)$, $v_{(C+C')^*}(S) = v_{C^*}(S) + v_{C'^*}(S)$ for all $S \subset N$. So, for each $G^l \in G$,*

$$\phi_l\left(M_0, \left(C + C'\right)^G\right) = \phi_l\left(M_0, C^G\right) + \phi_l\left(M_0, C'^G\right).$$

*By Lemma 1 $(b)$, for all $S \subset N$, $v_{C^*}(S \cup \{i\}) - v_{C^*}(S) = \min\limits_{j \in S_0}\{c^*_{ij}\}$. Therefore,*

$$\phi_k(M_0, C^G) \geq \min_{k' \in M_0 \backslash \{k\}}\{(c^G_{kk'})^*\}.$$

*Since the irreducible matrix is the minimal network associated with an mt,*

$$\min_{k' \in M_0 \backslash \{k\}}\{(c^G_{kk'})^*\} \geq \min_{k' \in M_0 \backslash \{k\}}\{c^G_{kk'}\}.$$

*Because of the definition of $(N_0, C, G)$,*

$$\min_{k' \in M_0 \backslash \{k\}}\{c^G_{kk'}\} \geq \max_{jj' \in G^k}\{c_{jj'}\}.$$

*A similar result can be obtained for $C'$. Now, it is easy to conclude that $t^* = t_{G^k} \cup \{(0, i_j)\}$ with $i_j \in G^k$ is an mt in $(G^k_0, C^\phi)$, $(G^k_0, C'^\phi)$. Hence, $(G^k_0, C^\phi)$ and $(G^k_0, C'^\phi)$ are under the conditions of RA. Moreover, $C^\phi + C'^\phi = \left(C + C'\right)^\phi$. Since $\varphi$ satisfies RA, for all $i \in G^k$,*

$$\begin{aligned} f_i^4\left(N_0, C + C', G\right) &= \varphi_i(G^k_0, (C + C')^\phi) \\ &= \varphi_i(G^k_0, C^\phi + C'^\phi) \\ &= \varphi_i\left(G^k_0, C^\phi\right) + \varphi_i\left(G^k_0, C'^\phi\right) \\ &= f_i^4\left(N_0, C, G\right) + f_i^4\left(N_0, C', G\right). \end{aligned}$$

*2. $f^4$ satisfies SYMA. Let $i,j \in G^k \in G$ be symmetric agents in $(N_0, C, G)$. By definition of $C^\phi$, $i$ and $j$ are symmetric agents in $\left(G_0^k, C^\phi\right)$. Since $\varphi$ satisfies SYM, $\varphi_i\left(G_0^k, C^\phi\right) = \varphi_j\left(G_0^k, C^\phi\right)$. Thus,*

$$f_i^4(N_0, C, G) = \varphi_i\left(G_0^k, C^\phi\right) = \varphi_j\left(G_0^k, C^\phi\right) = f_j^4(N_0, C, G).$$

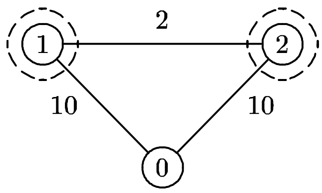
*3. $f^4$ satisfies PMC. Let $G^k \in G$. It is easy to prove that $\phi$ satisfies PM. Using arguments similar to those used in the proof of Claim 9 we can prove that $f^4$ satisfies PMC.*

*4. $f^4$ satisfies PMA.*

*We first prove that $\phi$ satisfies SCM. By Lemma 1 (b), for all $S \subset N$, $v_{C^*}(S \cup \{i\}) - v_{C^*}(S) = \min_{j \in S_0}\{c_{ij}^*\}$. Bergantiños and Vidal-Puga (2007a) prove that if $C \leq C'$, then $C^* \leq C'^*$. Now, it is easy to conclude that $\phi$ satisfies SCM.*

*Using arguments similar to those used in the proof of Claim 10 we can prove that $f^4$ satisfies PMA.*

*5. $f^4$ fails SYMC. Consider the mcstp with coalition structure where $N = \{1,2\}$, $G = \{G^1, G^2\}$, $G^1 = \{1\}$, $G^2 = \{2\}$ and matrix $C$ which is represented in the following figure:*




*Assume that $G^1$ comes before than $G^2$ in $\pi'$. Coalitions $G^1$ and $G^2$ are symmetric and $m(G^1, C) = m(G^2, C) = 0$. Nevertheless*

$$\begin{aligned} f_1^4(N_0, C, G) - m(G^1, C) &= \phi_1(M_0, C^G) = 10 \text{ and} \\ f_2^4(N_0, C, G) - m(G^2, C) &= \phi_2(M_0, C^G) = 2. \end{aligned}$$

■

***Claim 25*** *There exist rules satisfying SYMC, SYMA, PMC, and PMA but failing RA.*

***Proof.*** *We define the rule $f^5$ as follows. Let $(N_0, C, G)$ be an mcstp with coalition structure and $i \in G^k$.*

*We first define the rule $\sigma$ over $\left(M_0, C^G\right)$. Let $\Pi_G^e$ be the subset of permutations in which the coalitions with the expensive cost to the source connect first, i.e.*

$$\Pi_G^e = \left\{\pi \in \Pi_G \mid c_{0\pi(l)}^G \leq c_{0\pi(l')}^G \text{ when } \pi(l) > \pi(l')\right\}.$$

*For each $G^l \in G$, let $\sigma$ be the rule defined as*

$$\sigma_l\left(M_0, C^G\right) = \frac{1}{|\Pi_G^e|} \sum_{\pi \in \Pi_G^e} \left[ v_{(C^G)^*}\left(Pre\left(l,\pi\right) \cup \{l\}\right) - v_{(C^G)^*}\left(Pre\left(l,\pi\right)\right)\right].$$

*Now,*

$$f_i^5\left(N_0, C, G\right) = \varphi_i\left(G_0^k, C^\sigma\right)$$

*where*

$$c_{jj'}^\sigma = \begin{cases} c_{jj'} & \text{if } 0 \notin \{j,j'\} \\ \sigma_k\left(M_0, C^G\right) & \text{if } 0 \in \{j,j'\}. \end{cases}$$

*1. $f^5$ satisfies $SYMC$. It is trivial to see that $\sigma$ satisfies $SYM$. Using arguments similar to those used in the proof of Claim 7 we can prove that $f^5$ satisfies $SYMC$.*

*2. $f^5$ satisfies $SYMA$. Using arguments similar to those used in the proof of Claim 8 we can prove that $f^5$ satisfies $SYMA$.*

*3. $f^5$ satisfies $PMC$. Using arguments similar to those used in Bergantiños and Vidal-Puga (2007a), it is possible to prove that $\sigma$ satisfies $PM$. Using arguments similar to those used in the proof of Claim 9 we can prove that $f^5$ satisfies $PMC$.*

*4. $f^5$ satisfies $PMA$. Let $G^k \in G$ and $i \in G^k$, $G^k \neq \{i\}$. Let us denote as $C'$ the cost matrix $C$ restricted to the problem $((N \backslash \{i\})_0, C, (G \backslash G^k) \cup (G^k \backslash \{i\}))$ and $G' = (G \backslash G^k) \cup (G^k \backslash \{i\})$. We consider two cases:*

*a) Assume that $c_{kl}^G = c_{kl}'^{G'}$ for all $l \in M_0$. Thus, $\sigma_l\left(M_0, C^G\right) = \sigma_l\left(M_0, C'^{G'}\right)$ for all $l = 1, ..., m$. Hence, $\left((G^k \backslash \{i\})_0, C^\sigma\right) = \left((G^k \backslash \{i\})_0, C'^\sigma\right)$. Moreover, $\sigma$ satisfies $PM$. Using arguments similar to those used in the proof of Claim 10 we can prove that for all $j \in G^k \backslash \{i\}$,*

$$f_j^5(N_0, C, G) = f_j^5((N \backslash \{i\})_0, C, G')$$

*and for all $G^l \in G$, $l \neq k$ and all $j \in G^l$*

$$f_j^5(N_0, C, G) = f_j^5((N \backslash \{i\})_0, C', G').$$

*b) Assume that $c_{kk^*}^G \neq c_{kk^*}'^{G'}$ for some $k^* \in M_0$. Then, $c_{kk^*}^G < c_{kk^*}'^{G'}$. Moreover, $c_{0l}^G = c_{0l}'^{G'}$ for all $l \neq k$.*

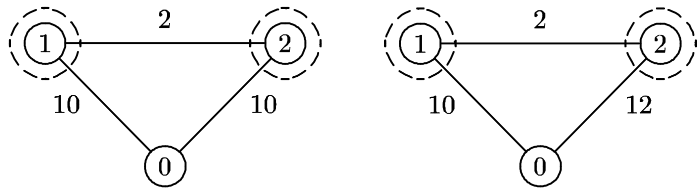
*This means that $\Pi_{G'}^e \subset \Pi_G^e$. Moreover, if $\pi \in \Pi_G^e$ and $\pi \notin \Pi_{G'}^e$ there exists $\pi' \in \Pi_{G'}^e$ such that $\pi'_{G \backslash G^k} = \pi_{G \backslash G^k}$ and $\pi'\left(k\right) < \pi\left(k\right)$. Intuitively, coalition $k$ comes first in the orders of $\Pi_{G'}^e$ than in the orders of $\Pi_G^e$.*

*By Lemma 1 $(c)$, for each cost matrix $C$, $v_{C^*}$ is a concave game. Making some computations it is possible to prove that $\sigma_k\left(M_0, C^G\right) \leq \sigma_k\left(M_0, C'^{G'}\right)$.*

*Now, using arguments similar to those used in the proof of Claim 10 we can prove that for all* $j \in G^k \backslash \{i\}$,

$$f_j^5(N_0, C, G) \leq f_j^5((N\backslash\{i\})_0, C, G').$$

*5.* $f^5$ *fails RA. Consider the mcstp with coalition structure where* $N = \{1,2\}$, $G = \{G^1, G^2\}$, $G^1 = \{1\}$, $G^2 = \{2\}$ *and matrices* $C$ *and* $C'$ *which are represented in the following figures:*




$(N_0, C, G)$ $(N_0, C', G)$

*If we take* $t = \{(0,1), (1,2)\}$ *we realize that* $C$ *and* $C'$ *are under the conditions of RA.*

*Now* $\Pi_G^e(C) = \{12, 21\}$, $\Pi_G^e(C') = \{21\}$, $\Pi_G^e(C + C') = \{21\}$. *Thus,* $f^5(N_0, C, G) = (6,6)$, $f^5(N_0, C', G) = (2, 10)$, *and* $f^5(N_0, C + C', G) = (4, 20)$. ■

*And the proof of Proposition 20 is completed.* ■

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